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AV-differential geometry: Poisson and Jacobi structures☆

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Abstract

Based on ideas of W.M. Tulczyjew, a geometric framework for a frame-independent formulation of different problems in analytical mechanics is developed. In this approach affine bundles replace vector bundles of the standard description and functions are replaced by sections of certain affine line bundles called AV-bundles. Categorial constructions for affine and special affine bundles as well as natural analogs of Lie algebroid structures on affine bundles (Lie affgebroids) are investigated. One discovers certain Lie algebroids and Lie affgebroids canonically associated with an AV-bundle which are closely related to affine analogs of Poisson and Jacobi structures. Homology and cohomology of the latter are canonically defined. The developed concepts are applied in solving some problems of frame-independent geometric description of mechanical systems.

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1. Introduction

While there is no doubt about the role of analytical mechanics in explaining many problems in a variety of physical topics, it is worth stressing that classical mechanics is by no means *passé*. It is still an open theory with several challenges and with an influence on both: physics and mathematics. The standard formulation of analytical mechanics in the language of differential geometry is based on geometrical objects of vector character. The vector bundle $\mathsf{T}M$ of tangent vectors is used as a space of infinitesimal (dynamical) configurations, the vector bundle T^*M of covectors plays the role of a phase space, and the Poisson bracket derived from the symplectic form serve in the Hamiltonian formulation of dynamics in which one uses the vector space (actually an algebra) of functions. However, there are situations where one finds difficulties while working with vector-like objects. Here we list some examples.

1. As the first example we describe the problems in the relativistic mechanics of a charged particle in the external electromagnetic field. The standard Lagrangian L is a function on the space of infinitesimal configurations $\mathsf{T}M : L(v) = -\langle eA, v\rangle + m\sqrt{g(v, v)}$, where A is the one-form representing the electromagnetic potential, m is the mass and e is the charge of the particle. The Lagrangian depends on the gauge of the first type. An electromagnetic potential is a connection in the principal bundle with the structure group $(\mathbb{R}, +)$ over the space–time. To obtain the one-form representing the potential one has to choose a section of the bundle (gauge). Changes in the gauge lead to changes in the Lagrangian. The gauge independent description is possible only when we use affine objects.
2. The configuration space (the space of events) for the inhomogenous formulation of time-dependent mechanics is the space–time M fibrated over the time $\mathbb{R}$. First-jets of this fibration form the infinitesimal configuration space. Since there is the distinguished vector field ∂_t on $\mathbb{R}$, the first-jets of the fibration over time can be identified with those vectors tangent to M which project on ∂_t. Such vectors form an affine subbundle of the tangent bundle $\mathsf{T}M$. The bundle V^*M, dual to the bundle of vectors which are vertical with respect to the fibration over time, is the phase space for the problem. The phase space carries a canonical Poisson structure, but Hamiltonian fields for this structure are vertical with respect to the projection on time, so they cannot describe the dynamics. In the standard formulation the distinguished vector field ∂_t is added to the Hamiltonian vector field to obtain the dynamics. This can be done correctly when the fibration over time is trivial, i.e. when $M = Q \times \mathbb{R}$. When the fibration is not trivial one has to choose a reference vector field that projects onto ∂_t. Changing the reference vector field means changing the Hamiltonian. To have the description of the dynamic being independent on the reference field one has to use affine objects.
3. Let us look on energy and momentum in the most classical case of Newtonian mechanics. The Newtonian space–time is a four-dimensional affine space N with an absolute time one-form $\tau \in (\mathsf{V}(N))^*$, which is a linear function on the model vector space $\mathsf{V}(N)$. The dynamics is usually described in a fixed inertial frame. The inertial frames are represented by vectors $u \in \mathsf{V}(N)$ such that $\tau(u) = 1$, i.e. u is the space–time velocity of an inertial observer associated with the inertial frame. The space of infinitesimal configurations, i.e.

positions and velocities, is $N \times E_1$, where $E_1 \subset \mathsf{V}(N)$ consists of vectors v satisfying $\tau(v) = 1$. Fixing an inertial frame u allows us to identify E_1 with $E_0 = \mathrm{Ker}(\tau)$ which is a vector subspace of $\mathsf{V}(N)$. Therefore we can define momenta as elements of $E_0^* = (\mathsf{V}(N))^*/\langle\tau\rangle$. The momentum transforms according to the formula $p' = p + f(u, u')$ while changing the inertial frame. The transformation of energy is also affine, so we cannot describe the dynamics in the frame-independent way as long as we keep representing the momentum as a vector object. We need an affine object to replace the usual covector. We can say that the covector in this case carries too much structure and we need additional physical information (i.e. an inertial frame) to use it properly. But even in a fixed inertial frame the standard description is not satisfactory, because the identification of E_1 with E_0 at the very beginning leads to the use of a wrong Poisson structure to generate equations of motion from the Hamiltonian. This is a situation similar to the previous example (cf. [6,11]).

Of course, the above list of problems is not complete. Our aim is to develop the geometric framework for correct approaches. The standard geometric constructions based on the algebra of functions on a manifold M are replaced by constructions based on the affine space of sections of an affine bundle $\zeta : \mathbf{Z} \to M$, modeled on the trivial bundle $M \times \mathbb{R}$. Such an affine bundle we will call a *bundle of affine values* (*AV-bundle* in short). The elements of the bundle $\mathbf{Z}$ replace number-values of functions but we are not informed now what and where is zero for these values, so our “functions” do not form any algebra or even a vector space. Such an approach forces deep changes in the language, notions and canonical objects of differential geometry. We propose to call this kind of geometry *the differential geometry of affine values* (*AV-differential geometry* in short).

An additional motivation comes from the observation that even canonical objects in the traditional “vector geometry” happen to have an affine character, more or less hidden or forgotten. Let us consider the canonical symplectic form on the cotangent bundle T^*M. This 2-form is recognized as a linear object while, on the other hand, it is invariant with respect to translations by closed forms on M that suggests its hidden affine character. Indeed, it is possible to construct an affine analog of T^*M, which is a symplectic manifold with canonical symplectic structure and which seems to be more appropriate phase space for many mechanical problems.

The idea of using affine bundles for the correct frame-independent geometric formulation of analytical mechanics theories goes back to some concepts of Tulczyjew [21] (see also [1,22,24]). We will also use in the paper some of unpublished ideas of W.M. Tulczyjew. A similar approach to time-dependent non-relativistic mechanics (in the Lagrange formulation) has been recently developed by Massa et al. [15,16,26], Martínez et al. [12–14]. Our paper is organized as follows.

In Section 2 we present basic notions of the theory of affine spaces and relations to the theories of special (resp., cospecial) vector spaces, i.e. the vector spaces with distinguished a non-zero vector (resp., covector).

Basic categorial construction for affine spaces and special/cospecial vector spaces, like direct sums, products, and tensor products, are presented in Section 3. To our surprise, we could not find them in the literature.

In Section 4, the main affine objects of our approach, namely *special affine spaces*, i.e. affine spaces modeled on special vector spaces are introduced together with the corresponding notion of special duality.

One-dimensional special affine spaces, called *spaces of affine scalars* are of particular interest. Some properties of such spaces are investigated in Section 5.

All above is extended to the case of bundles in Sections 6 and 7. A *special affine bundle* is a pair $\mathbf{A} = (A, v)$, where A is an affine bundle over M and $v \in \mathsf{Sec}(\mathsf{V}(A))$ is a nowhere-vanishing section of its model bundle $\mathsf{V}(A)$. The *dual special affine bundle* $\mathbf{A}^\#$ is the affine bundle $\mathbf{Aff}(\mathbf{A}; \mathbf{I})$ of special affine morphisms of $\mathbf{A}_m$ into the canonical special affine bundle $M \times \mathbf{I}$, where $\mathbf{I} = (\mathbb{R}, 1)$, i.e. those affine morphisms $\varphi : A_m \to \mathbb{R}$ whose linear part maps $v(m)$ into 1, $m \in M$, with the distinguished section of the model vector bundle being 1_A—the constant 1 function on A.

One-dimensional special affine bundles are called *AV-bundles*. An important observation is that there is a one-to-one correspondence between the space $\mathsf{Sec}(\mathbf{A})$ of sections of a special affine bundle $\mathbf{A}$ and the space $\mathsf{Aff}\,\mathsf{Sec}(\mathbf{A}^\#)$ of affine sections of the bundle $\mathbf{A}^\# \to \mathbf{A}^\#/\langle 1_A \rangle$ which is canonically an AV-bundle. The affine sections are, of course, those sections $\sigma : \mathbf{A}^\#/\langle 1_A \rangle \to \mathbf{A}^\#$ which are affine maps, i.a. morphisms of affine bundles. This is a special affine analog of the well-known correspondence between sections of a vector bundle E and linear functions on the dual bundle E^*.

In Section 8 the phase $\mathsf{P}\mathbf{Z}$ and the contact bundle $\mathsf{C}\mathbf{Z}$ associated with an AV-bundle $\mathbf{Z}$ are constructed. They are AV-analogs of T^*M and $\mathsf{T}^*M \oplus \mathbb{R}$ and carry canonical symplectic and contact structures, respectively. The AV-Liouville one-form which is the potential of the canonical symplectic form on $\mathsf{P}\mathbf{Z}$ is naturally understood as a section of an affine fibration over $\mathsf{P}\mathbf{Z}$ (cf. [25]).

Various Lie algebroids and Lie affgebroids (i.e. Lie algebroid-like objects on affine bundles [2]) associated with a given AV-bundle $\mathbf{Z}$ are defined and studied in Sections 9–11. Let us mention the Lie algebroid $\tilde{\mathsf{T}}\mathbf{Z}$ (an AV-analog of the Lie algebroid extension $\mathsf{L}M = \mathsf{T}M \oplus \mathbb{R}$ of the canonical Lie algebroid $\mathsf{T}M$ of vector fields), the Lie algebroid $\tilde{\mathsf{L}}\mathbf{Z}$ (an AV-analog of the Lie algebroid extension $\mathsf{L}M \oplus \mathbb{R}$ of the Lie algebroid $\mathsf{L}M = \mathsf{T}M \oplus \mathbb{R}$ of linear first-order differential operators on M) and their affine counterparts $\bar{\mathsf{T}}\mathbf{Z}$ and $\bar{\mathsf{L}}\mathbf{Z}$. One proves that the Lie algebroid $\tilde{\mathsf{L}}\mathbf{Z}$ admits a canonical closed one-form ϕ^0, i.e. $\tilde{\mathsf{L}}\mathbf{Z}$ carries a canonical structure of a *Jacobi algebroid* (see [4,5,10]). It is also shown that sections of $\tilde{\mathsf{T}}\mathbf{Z}$, or $\tilde{\mathsf{L}}\mathbf{Z}$, (resp., sections of $\bar{\mathsf{T}}\mathbf{Z}$, or $\bar{\mathsf{L}}\mathbf{Z}$) can be interpreted as affine derivations, or affine first-order differential operators, on sections of $\mathbf{Z}$ with values in functions on M (resp., such derivations, or first-order differential operators, but with values in sections on $\mathbf{Z}$).

In Section 12 we recall the definitions and basic facts on aff-Poisson and aff-Jacobi brackets (cf. [2]), i.e. analogs of Poisson and Jacobi brackets, defined on sections of an AV-bundle $\mathbf{Z}$ over M and taking values in the ring of smooth functions $C^\infty(M)$. The main result is the correspondence between Lie affgebroid structures on a special affine bundle $\mathbf{A} = (A, v)$ and aff-Jacobi brackets on the AV-bundle $\mathbf{A}^\# \to \mathbf{A}^\#/\langle 1_A \rangle$ which are affine in the sense that the bracket of two affine sections is an affine function on $\mathbf{A}^\#/\langle 1_A \rangle$. This can be viewed as an AV-analog of the fact that Lie algebroid brackets on a vector bundle E correspond to linear Poisson brackets on the dual bundle E^*. In this picture, the Lagrange formulation of a mechanical problem takes place on a special affine bundle $\mathbf{A} = (A, v)$

equipped with a Lie affgebroid structure, and the Lagrangians are sections of the AV-bundle $\mathbf{A} \to \mathbf{A}/\langle v \rangle$. The Hamilton formalism, in turn, takes place on the dual special affine bundle $\mathbf{A}^{\#}$ and the Hamiltonians are sections of the AV-bundle $\mathbf{A}^{\#} \to \mathbf{A}^{\#}/\langle 1_{\mathcal{A}} \rangle$ which carries a canonical aff-Jacobi structure. In most important examples this structure happens to be aff-Poisson.

In Section 13 we observe that aff-Poisson and aff-Jacobi structures on an AV-bundle $\mathbf{Z}$ correspond to *canonical structures* Λ and $\mathcal{J}$ for the Lie algebroid $\tilde{\mathsf{T}}\mathbf{Z}$ and Jacobi algebroid $\tilde{\mathsf{L}}\mathbf{Z}$, respectively, i.e. $\Lambda \in \wedge^2 \tilde{\mathsf{T}}\mathbf{Z}$, $[\![\Lambda, \Lambda]\!]_{\tilde{\mathsf{T}}\mathbf{Z}} = 0$ (resp., $\mathcal{J} \in \wedge^2 \tilde{\mathsf{L}}\mathbf{Z}$, $[\![\mathcal{J}, \mathcal{J}]\!]^{\phi^0}_{\tilde{\mathsf{L}}\mathbf{Z}} = 0$), where $[\![\cdot, \cdot]\!]_{\tilde{\mathsf{T}}\mathbf{Z}}$ is the Lie algebroid Schouten bracket on $\wedge^\bullet \tilde{\mathsf{T}}\mathbf{Z}$ (resp., $[\![\cdot, \cdot]\!]^{\phi^0}_{\tilde{\mathsf{L}}\mathbf{Z}}$ is the Schouten–Jacobi bracket of the Jacobi algebroid $(\tilde{\mathsf{L}}\mathbf{Z}, \phi^0)$). This is an AV-analog of the well-known identification of Poisson brackets on $C^\infty(M)$ with Poisson tensors on M, i.e. bivector fields with the Schouten–Nijenhuis square being 0. The known results on characterization of canonical structures for Lie and Jacobi algebroids [5,9] allow one to derive an analogous characterization for aff-Poisson and aff-Jacobi brackets. In particular, one can define the corresponding homology and cohomology in a natural way.

In Section 14 we present solutions of the mentioned problems of the frame-independent geometric formulation in analytical mechanics with the use of developed concepts. These solutions form an alternative to the Kaluza–Klein approach where the vector-like formulations is kept for the price of extending the dimension (see also [17,20,21].

Much of this material is to our knowledge new. Our aim was to present a possibly complete picture which can be viewed as a well-described mathematical program based on the ideas and needs from analytical mechanics.

2. Category of affine spaces

An *affine space* is a triple (A, V, α), where A is a set, V is a vector space over a field $\mathbb{K}$ and α is a mapping $\alpha : A \times A \to V$ such that

- $\alpha(a_3, a_2) + \alpha(a_2, a_1) + \alpha(a_1, a_3) = 0$;
- the mapping $\alpha(\cdot, a) : A \to V$ is bijective for each $a \in A$.

We shall also write simply A to denote the affine space (A, V, α) and $\mathsf{V}(A)$ to denote V. One can also say that an affine space is a set with a free and transitive action of a vector space (which is viewed as a commutative group with respect to addition). By *dimension* of A we understand the dimension of $\mathsf{V}(A)$. If (A, V, α) then also $(A, V, -\alpha)$ is an affine space. We will write for brevity $\bar{A}^a$ to denote the *adjoint affine space* $(A, V, -\alpha)$. We will write also $a_2 - a_1$ instead of $\alpha(a_2, a_1)$ and $a + v$ to denote the unique point $a' \in A$ such that $a' - a = v$, $v \in \mathsf{V}(A)$. Of course, every vector space is canonically an affine space modeled on itself with the affine structure $\alpha(v_1, v_2) = v_1 - v_2$. The adjoint affine space $\bar{A}^a$ can be viewed as the same set A with the opposite action of $\mathsf{V}(A)$: $a \mapsto a - v$.

It is easy to see that for any linear subspace V_0 of V the set A/V_0 of cosets of A with respect to the relation $a \sim a' \Leftrightarrow a - a' \in V_0$ is canonically an affine space modeled on V/V_0.

A subset A' of A is an *affine subspace* in A if there is a linear subspace $\mathsf{V}(A')$ of $\mathsf{V}(A)$ such that $A' = a' + \mathsf{V}(A')$ for certain $a' \in A'$. Affine subspaces are canonically affine spaces with the affine structure inherited from A.

If A' is an affine subspace of A then the quotient space A/A' is understood as $A/\mathsf{V}(A')$ with distinguished point being the class of A'. Hence A/A' can be identified with the linear space $\mathsf{V}(A)/\mathsf{V}(A')$.

Morphisms in the category of affine spaces are affine maps. Let A and A' be affine spaces. We say that a mapping $\varphi : A \to A'$ is *affine* if there is a linear mapping $\varphi_{\mathsf{V}} : V \to V'$ such that

$$\varphi(a + v) = \varphi(a) + \varphi_{\mathsf{V}}(v).$$

We say that φ_{V} is the *linear part* of φ.

More generally, on every affine space instead of the subtraction $a_1 - a_2$, one can consider *vector combination* of elements of A, i.e. the combination $\sum_i \lambda_i a_i$, where $a_i \in A$, $\lambda_i \in \mathbb{K}$, and $\sum_i \lambda_i = 0$. Every vector combination of elements of A defines a unique element of $\mathsf{V}(A)$ in obvious way. Similarly, one can consider *affine combinations* (called also *barycentric combinations*) of elements of A which have formally the same form but with $\sum_i \lambda_i = 1$. An affine combination determines uniquely an element of A. Affine maps may be equivalently defined as those maps which respect affine combinations. Note however that affine combinations do not determine the affine structure completely: A and $\bar{A}^a$ have the same affine combinations. The set $\mathsf{Aff}(A; A')$ of all affine maps from A into A' is again an affine space modeled on the vector space $\mathsf{Aff}(A; \mathsf{V}(A'))$ of affine maps from A into the model vector space $\mathsf{V}(A')$ of A': for $\varphi_1, \varphi_2 \in \mathsf{Aff}(A_1; A_2)$ we put $(\varphi_1 - \varphi_2)(a) = \varphi_1(a) - \varphi_2(a)$. Inductively, the set $\mathsf{Aff}^k(A_1, \dots, A_k; A)$ of k-affine maps from $A_1 \times \cdots \times A_k$ into A is defined as the set $\mathsf{Aff}(A_1; \mathsf{Aff}^{k-1}(A_2, \dots, A_k; A))$. Like in the linear case, one proves that $\mathsf{Aff}^k(A_1, \dots, A_k; A)$ can be identified with the space of of maps $F : A_1 \times \cdots \times A_k \to A$ which are affine with respect to every variable separately. By

$$F^i_{\mathsf{V}} : A_1 \times \cdots \times \mathsf{V}(A_i) \times \cdots \times A_k \to \mathsf{V}(A)$$

we denote the linear part of F with respect to the ith variable. It is linear on $\mathsf{V}(A_i)$ and affine with respect to each of the remaining variables separately. The higher-order linear parts $F^{i_1,\dots,i_l}_{\mathsf{V}}$ are defined in obvious way. The multilinear map

$$F^{1,\dots,k}_{\mathsf{V}} : \mathsf{V}(A_1) \times \cdots \times \mathsf{V}(A_k) \to \mathsf{V}(A)$$

we denote simply by F_v.

A *free affine space* $\mathcal{A} = \mathcal{A}(\{a_j\}_{j\in J})$ generated by the set $\{a_j : j \in J\}$ is an affine subspace in the free vector space generated by $\{a_j : j \in J\}$, i.e. in the the vector space $\mathcal{V}(\{a_j\}_{j\in J})$ of formal linear combinations $v = \sum_j \lambda_j a_j$, described by the equation $1_{\mathcal{A}}(v) = 1$, where $1_{\mathcal{A}}$ is the linear functional on $\mathcal{V}(\{a_j\}_{j\in J})$ defined by $1_{\mathcal{A}}(v) = \sum_j \lambda_j$. The notation is justified by the fact that this functional is constantly 1 on $\mathcal{A}$. The model vector space for this free affine space is a linear subspace of $\mathcal{V}(\{a_j : j \in J\})$ being the kernel of the functional $1_{\mathcal{A}}$. Every affine space A is actually isomorphic to the free affine space generated by a subset of A which we call a *basis* of A. A subset $\{a_j : j \in J\}$ is a basis of A if every element of A can be expressed uniquely as an affine combination of elements of the basis. Existence of a basis can be proved analogously to the linear case, since $\{a_j : j \in J\}$ is a basis of A if and only if $\{a_j - a_{j_0} : j \in J'\}$ is a basis of $\mathsf{V}(A)$, where $j_0 \in J$ and $J' = J \setminus \{j_0\}$. The dimension of A is the cardinality of a basis minus 1.

Every affine space A is canonically embedded as an affine hyperspace into a vector space $\hat{A}$ which we call the *vector hull* of A. The vector hull $\hat{A}$ is defined as the quotient space $\mathcal{V}(A)/\mathcal{V}_0(A)$ of the free vector space $\mathcal{V}(A)$ generated by A by its subspace spanned by linear combinations of the form $1 \cdot (a + \lambda(a' - a'')) - 1 \cdot a - \lambda a' + \lambda a''$. Here the expression $(a + \lambda(a' - a''))$ is viewed as an element of A. Since A is canonically embedded into $\mathcal{V}(A)$ as a set, we have a canonical map from A into $\hat{A}$ which can be proved to be an embedding of the affine space onto an *affine hyperspace*, i.e. a one-codimensional affine subspace which is proper (does not contain 0), of $\hat{A}$. This hyperspace can be equivalently defined as the level-1 set of the functional $1_A : \hat{A} \to \mathbb{K}$ represented by the sum of coefficients on $\mathcal{V}(A)$. We will not denote this embedding in a special way just regarding A as a subset of $\hat{A}$. The model vector space $\mathsf{V}(A)$ is also canonically embedded in $\hat{A}$ as the kernel of 1_A.

Choosing a basis $\{a_j : j \in J\}$ of A we get an isomorphism of $\hat{A}$ with $\mathcal{V}(\{a_j : j \in J\})$. Note that for a vector space V viewed as an affine space its vector hull $\hat{V}$ is canonically isomorphic to $V \oplus \mathbb{K}$. This decomposition follows from the existence of a distinguished element $0 \in V$ which is a non-zero vector in $\hat{V}$ complementary to $\mathsf{V}(V) \simeq V$. It is obvious by construction that the vector hull is unique up to isomorphism, so that we have the following theorem.

Theorem 1. *Every affine space A is canonically embedded as an affine hyperspace of the vector space $\hat{A}$—its vector hull. Conversely, if A is embedded as an affine hyperspace of a vector space W, then there is a canonical isomorphism $\Phi : \hat{A} \to W$ which reduces to the identity map on the embedded A.*

For vector spaces V_1, V_2 we denote by $\mathsf{Hom}(V_1; V_2)$ the space of morphisms (linear maps) from V_1 into V_2 and by $\mathsf{Hom}_{A_1}^{A_2}(V_1; V_2)$ the subset of those morphisms $\Phi \in \mathsf{Hom}(V_1; V_2)$ which map the subset A_1 of V_1 into the subset A_2 of V_2.

Theorem 2. *For an affine space A and a vector space V there are canonical identifications*

(a) $\mathsf{Aff}(A, V) \ni \varphi \mapsto \hat{\varphi} \in \mathsf{Hom}(\hat{A}, V)$.
In particular, the vector space $A^\dagger = \mathsf{Aff}(A, \mathbb{R})$ is canonically isomorphic to $\hat{A}^*$, and

(b) $\mathsf{Aff}(A_1, A_2) \ni \varphi \mapsto \hat{\varphi} \in \mathsf{Hom}_{A_1}^{A_2}(\hat{A}_1, \hat{A}_2)$
for affine spaces A_1, A_2.

Proof. We put simply $\hat{\varphi}(\sum_i \lambda_i a_i) = \sum_i \lambda_i \varphi(a_i)$ for $\lambda_i \in \mathbb{K}$, $a_i \in A$, so $\varphi = \hat{\varphi}_{|A}$. There are obvious embeddings

$$\mathsf{Aff}(A_1, A_2) \subset \mathsf{Aff}(A_1, \hat{A}_2) \subset \mathsf{Hom}(\hat{A}_1, \hat{A}_2)$$

and it is easy to see that $\mathsf{Aff}(A_1, A_2)$ is characterized inside $\mathsf{Hom}(\hat{A}_1, \hat{A}_2)$ as the set of those morphisms which map A_1 into A_2. □

The vector space $\hat{A}$ has a distinguished affine hyperspace A. Such an affine subspace is uniquely determined by the nonzero functional $1_A \in \hat{A}^*$ as its level-1 set: $A = \{v \in \hat{A} : 1_A(v) = 1\}$. Thus $\hat{\mathbf{A}} = (\hat{A}, 1_A)$ is an example of a *cospecial vector space*, i.e. a vector space with a distinguished affine hyperspace, or, equivalently, as a vector space with a distinguished non-zero linear functional. On the other hand, its vector dual $\hat{A}^*$, which is canonically identified with $A^\dagger = \mathsf{Aff}(A, \mathbb{K})$, is a *special vector space*, i.e. a vector space with a distinguished non-zero element. We will denote this special vector space by $\mathbf{A}^\dagger = (A^\dagger, 1_A)$ and call it the *vector dual* of A. In finite dimension we have a true duality between affine spaces and special vector spaces. Indeed, every special vector space $\mathbf{V} = (V, v^0)$ defines an affine hyperspace $\mathbf{V}^\ddagger = \{u \in V^* : u(v^0) = 1\}$ in the dual V^* of V. Since in finite dimension $(V^*)^* = V$, we have the following theorem.

Theorem 3. *For finite-dimensional special vector space* $\mathbf{V}$ *and finite-dimensional affine space* A *there are canonical isomorphisms*

$$((\mathbf{V}^\ddagger)^\dagger, 1_{\mathbf{V}^\ddagger}) \simeq \mathbf{V}$$

and

$$(\mathbf{A}^\dagger)^\ddagger \simeq A.$$

The vector hull $\widehat{\mathsf{Aff}}(A_1, A_2)$ of $\mathsf{Aff}(A_1, A_2)$ can be interpreted as the vector space $\widehat{\mathsf{Hom}}(\hat{\mathbf{A}}_1, \hat{\mathbf{A}}_2)$ of those linear maps $F : \hat{A}_1 \to \hat{A}_2$ for which $F^*(1_{A_2}) = \lambda 1_{A_1}$ for certain $\lambda \in \mathbb{K}$.

Special (resp., cospecial) vector spaces form a category with the set of morphisms $\mathbf{Hom}(\mathbf{V}_1, \mathbf{V}_2)$ between $\mathbf{V}_i = (V_i, v_i^0)$ (resp. $\mathbf{V}_i = (V_i, \varphi_i)$), $i = 1, 2$, consisting of those linear maps $F : V_1 \to V_2$ for which $F(v_1^0) = v_2^0$ (resp., $F^*(\varphi_2) = \varphi_1$). The condition $F^*(\varphi_2) = \varphi_1$ means that F maps the points of the affine hyperspace $A_1 = \{\varphi_1(u_1) = 1\}$ of V_1 into the affine hyperspace $A_2 = \{\varphi_2(u_2) = 1\}$ of V_2. There is a canonical covariant equivalence functor from the category of cospecial vector spaces into the category of affine spaces. It associates with any cospecial vector space (V, A) its affine hyperspace A, and with every morphism $F : (V_1, A_1) \to (V_2, A_2)$ its restriction to A_1 (which is an affine map into A_2). Conversely, with every affine space A we associate its vector hull $\hat{A}$ with A as the distinguished affine hyperspace and with every affine map $F : A_1 \to A_2$ its (unique) extension to a linear map from $\hat{A}_1$ into $\hat{A}_2$. In finite dimensions we can use the duality and obtain a contravariant equivalence functor from the category of special vector spaces to the category of affine spaces. This functor associates with a special vector space $\mathbf{V} = (V, v^0)$ the affine hyperspace $\mathbf{V}^\ddagger$ in V^*. We will use these equivalences to construct categorial object for the affine category exploring (generally better) knowledge of the linear category.

3. Categorial constructions for affine spaces

In the category of special vector spaces and, consequently, in the category of cospecial vector spaces and the category of affine spaces there are direct sums and products.

We will just describe the models leaving the obvious proofs to the reader. The constructions are very natural but, to our surprise, we could not find explicit references in the literature.

For special vector spaces $\mathbf{V}_i = (V_i, v_i^0)$, $i = 1, 2$, their product $\mathbf{V}_1 \times^{sv} \mathbf{V}_2$ is represented by the standard product $V_1 \times V_2$ with the distinguished vector $v^0 = (v_1^0, v_2^0)$. The projections $\pi_i : V_1 \times V_2 \to V_i$ map v^0 onto v_i^0, i.e. represent morphisms of special vector spaces.

The *special direct* sum $\mathbf{V}_1 \oplus^{sv} \mathbf{V}_2$ is represented by the quotient vector space $V_1 \oplus V_2 / \langle v_1^0 - v_2^0 \rangle$ with the distinguished vector being the class $[v_1^0]$ of v_1^0 (or, equivalently, the class $[v_2^0]$ of v_2^0). The embedding of $\mathbf{V}_i$ is represented by the embedding of V_i in $V_1 \oplus V_2$ composed with the projection.

By duality, for cospecial vector spaces $\mathbf{V}_i = (V_i, \varphi_i^0)$, $i = 1, 2$, its *cospecial direct sum* $\mathbf{V}_1 \oplus^{cv} \mathbf{V}_2$ is represented by the vector space $V_1 \oplus V_2$ with the distinguished functional $\varphi^0 = (\varphi_1^0, \varphi_2^0) \in V_1^* \times V_2^* = (V_1 \oplus V_2)^*$ and obvious embeddings of $\mathbf{V}_i$. The product $\mathbf{V}_1 \times^{cv} \mathbf{V}_2$, in turn, is represented by the linear hyperspace in $V_1 \times V_2$ being the kernel of $\varphi_1^0 - \varphi_2^0 \in V_1^* \oplus V_2^* = (V_1 \times V_2)^*$ and equipped with the distinguished functional $\varphi^0 = (\varphi_1^0)_{|\mathrm{Ker}(\varphi_1^0 - \varphi_2^0)} = (\varphi_2^0)_{|\mathrm{Ker}(\varphi_1^0 - \varphi_2^0)}$. The projections from $\mathrm{Ker}(\varphi_1^0 - \varphi_2^0)$ onto V_i are just restrictions of projections from $V_1 \times V_2$. They give rise to cospecial morphisms from $\mathbf{V}_1 \times^{cv} \mathbf{V}_2$ onto $\mathbf{V}_i$.

The above constructions allow us to recognize the products and sums in the category of affine spaces. The *affine product* $A_1 \times^a A_2$ in this category is the standard Cartesian product $A_1 \times A_2$ which is an affine space modeled on $\mathsf{V}(A_1) \times \mathsf{V}(A_2)$, $(a_1, a_2) - (a_1', a_2') = (a_1 - a_1', a_2 - a_2')$. The direct sum $A_1 \oplus^a A_2$ is the affine hyperspace in $\hat{A}_1 \oplus \hat{A}_2$ generated by the affine subspaces A_1, A_2 which are canonically embedded, i.e.

$$A_1 \overset{a}{\oplus} A_2 = \{\lambda_1 a_1 + \lambda_2 a_2 \in \hat{A}_1 \oplus \hat{A}_2 : a_1 \in A_1, a_2 \in A_2, \lambda_1 + \lambda_2 = 1\},$$

with obvious embeddings of A_1 and A_2.

Theorem 4. *We have canonical isomorphisms*

$$\widehat{(A_1 \overset{a}{\times} A_2)} \simeq \hat{A}_1 \overset{cv}{\times} \hat{A}_2, \tag{1}$$

$$\widehat{(A_1 \overset{a}{\oplus} A_2)} \simeq \hat{A}_1 \overset{cv}{\oplus} \hat{A}_2, \tag{2}$$

$$(A_1 \overset{a}{\times} A_2)^\dagger \simeq A_1^\dagger \overset{sv}{\oplus} A_2^\dagger, \tag{3}$$

$$(A_1 \overset{a}{\oplus} A_2)^\dagger \simeq A_1^\dagger \overset{sv}{\times} A_2^\dagger, \tag{4}$$

where the vector hulls and the vector duals are regarded as cospecial and special vector spaces, respectively.

In the category of affine spaces we can define *affine tensor products* $A_1 \otimes^a \cdots \otimes^a A_k$ which are affine spaces such that $\mathsf{Aff}^k(A_1, \ldots, A_k; A) = \mathsf{Aff}(A_1 \otimes^a \cdots \otimes^a A_k; A)$. Like in the linear case, $A_1 \otimes^a \cdots \otimes^a A_k$ can be defined as the quotient of the free affine space

$\mathcal{A}(A_1 \times \cdots \times A_k)$ by the linear subspace of its model vector space generated by elements of the form

$$(a_1, \ldots, a_i + \lambda(a_i' - a_i''), \ldots, a_k) - (a_1, \ldots, a_i, \ldots, a_k) - \lambda(a_1, \ldots, a_i', \ldots, a_k) + \lambda(a_1, \ldots, a_i'', \ldots, a_k).$$

One can also say that $A_1 \otimes^a \cdots \otimes^a A_k$ is the affine subspace in $\hat{A}_1 \otimes \cdots \otimes \hat{A}_k$ spanned by tensors of the form $a_1 \otimes \cdots \otimes a_k$, where $a_i \in A_i$. The tensor product $A_1 \otimes^a \cdots \otimes^a A_k$ may be viewed also as the affine hyperspace in the standard tensor product $\hat{A}_1 \otimes \cdots \otimes \hat{A}_k$ determined by the functional $1_{A_1} \otimes \cdots \otimes 1_{A_k}$ and the associated vector space $\mathsf{V}(A_1 \otimes^a \cdots \otimes^a A_k)$ is the kernel of $1_{A_1} \otimes \cdots \otimes 1_{A_k}$. It is easy to see that $\mathsf{V}(A_1 \otimes^a \cdots \otimes^a A_k)$ is additively generated by tensors $a_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes a_k$ from $\hat{A}_1 \otimes \cdots \otimes \hat{A}_k$, where $a_j \in A_j$, $v_i \in \mathsf{V}(A_i)$. This is indeed a vector space, since $\lambda(a_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes a_k)$ is represented by $a_1 \otimes \cdots \otimes \lambda v_i \otimes \cdots \otimes a_k$. If we fix $a_i^0 \in A_i$, then

$$A_1 \overset{a}{\otimes} \cdots \overset{a}{\otimes} A_k = a_1^0 \otimes \cdots \otimes a_k^0 + \bigoplus_{i_1 < \cdots < i_l} a_1^0 \otimes \cdots \otimes \mathsf{V}(A_{i_1}) \otimes \cdots \otimes \mathsf{V}(A_{i_l}) \otimes \cdots \otimes a_k^0.$$

Sometimes we will write $a_1 \otimes^a \cdots \otimes^a a_k$ for the affine tensor product represented by $a_1 \otimes \cdots \otimes a_k \in \hat{A}_1 \otimes \cdots \otimes \hat{A}_k$ to stress that we are dealing with an element of $A_1 \otimes^a \cdots \otimes^a A_k$. The canonical map

$$A_1 \times \cdots \times A_k \ni (a_1, \ldots, a_k) \mapsto a_1 \overset{a}{\otimes} \cdots \overset{a}{\otimes} a_k \in A_1 \overset{a}{\otimes} \cdots \overset{a}{\otimes} A_k$$

is a multi-affine map. Note that for vector spaces V_i there is a canonical identification of $V_1 \otimes^a \cdots \otimes^a V_k$ with $((V_1 \oplus \mathbb{K}) \otimes \cdots \otimes (V_k \oplus \mathbb{K})) \ominus (\mathbb{K} \otimes \cdots \otimes \mathbb{K})$. For the dimension we have the formula

$$\dim(A_1 \overset{a}{\otimes} \cdots \overset{a}{\otimes} A_k) = (\dim(A_1) + 1) \cdots (\dim(A_k) + 1) - 1.$$

Like in the linear case, we have natural isomorphisms

$$A_1 \overset{a}{\otimes} A_2 \simeq A_2 \overset{a}{\otimes} A_1, \tag{5}$$

$$A_1 \overset{a}{\otimes} (A_2 \overset{a}{\otimes} A_3) \simeq (A_1 \overset{a}{\otimes} A_2) \overset{a}{\otimes} A_3, \tag{6}$$

$$\widehat{A_1 \overset{a}{\otimes} A_2} \simeq \hat{A}_1 \otimes \hat{A}_2, \tag{7}$$

$$(A_1 \overset{a}{\otimes} A_2)^\dagger \simeq (A_1)^\dagger \otimes (A_2)^\dagger. \tag{8}$$

To define affine skew-symmetric tensor product $(\wedge^a)^k A$ for $k > 1$, let us observe first that the symmetric group S_k acts naturally on $A^{\otimes^a k}$. By A_0^k we denote its affine subspace spanned by tensors of the form $a_1 \otimes^a \cdots \otimes^a a_k$, where $a_i = a_j$ for certain $i \neq j$, i.e. invariant with respect to a transposition. We put

$$(\overset{a}{\bigwedge})^k A = A^{\otimes^a k} / A_0^k$$

which is canonically a vector space (see the previous section). It follows directly from definition that any element of $\mathsf{Aff}((\wedge^a)^k A; A')$ represents a multi-affine mapping $F : A \times \cdots \times A \to A'$ (an element of $\mathsf{Aff}(A^{\otimes^a k}; A'))$ which is constant on A_0^k, i.e. constant on the set of those $(a_1, \ldots, a_k)$ for which $a_i = a_j$ for certain $i \neq j$.

It is a standard task to prove that such multi-affine mappings are *skew-symmetric* in the sense that

$$F_{\mathsf{V}}^{\sigma^{-1}(1)}((a_{\sigma(1)}, \ldots, v, \ldots, a_{\sigma(k)}) = \operatorname{sgn}(\sigma) F_{\mathsf{V}}^1(v, a_2, \ldots, a_k)$$

for any permutation $\sigma \in S_k$ and $a_2, \ldots, a_k \in A$, $v \in \mathsf{V}(A)$. It is easily seen that, for $k > 1$, the affine wedge product $(\wedge^a)^k A$ is canonically isomorphic to the standard exterior power $\wedge^k \hat{A}$. To put it simpler, one can also say that $(\wedge^a)^k A$ is the affine subspace in $\wedge^k \hat{A}$ generated by tensors $a_1 \wedge \cdots \wedge a_k$, $a_i \in A$, which happens to be the whole $\wedge^k \hat{A}$.

4. Special affine spaces and special duality

A *special affine space* $\mathbf{A} = (A, v^0)$ is an affine space A modeled on a special vector space $\mathbf{V}(A) = (\mathsf{V}(A), v^0)$. The *adjoint special affine space* $\bar{\mathbf{A}} = (A, -v^0)$ is modeled on the *adjoint special vector space* $\bar{\mathbf{V}} = (V, -v^0)$.

Let $\mathbf{A} = (A, v^0)$ and $\mathbf{A}_i = (A_i, v_i^0)$, $i = 1, \ldots, k$, be special affine spaces with the distinguished vectors $v^0 \in \mathbf{V}(A)$, $v_i^0 \in \mathbf{V}(A_i)$. By $\mathbf{Aff}(\mathbf{A}_1; \mathbf{A})$ we denote the space of special affine maps $\varphi : \mathbf{A}_1 \to \mathbf{A}$. It is canonically a special affine space, since the constant map onto $\{v^0\}$ is naturally distinguished in $\mathsf{V}(\mathsf{Aff}(A_1; A)) = \mathsf{Aff}(A_1, \mathsf{V}(A))$. Inductively, we put

$$\mathbf{Aff}^k(\mathbf{A}_1, \ldots, \mathbf{A}_k; \mathbf{A}) = \mathbf{Aff}(\mathbf{A}_1; \mathbf{Aff}^{k-1}(\mathbf{A}_2, \ldots, \mathbf{A}_k; \mathbf{A}))$$

for the space of k-special affine maps from $\mathbf{A}_1 \times \cdots \times \mathbf{A}_k$ into $\mathbf{A}$, which consists of maps $F : A_1 \times \cdots \times A_k \to A$ which are special affine with respect to every variable separately.

The vector hull of a special affine space is canonically a *bispecial vector space*, i.e. a vector space V with a distinguished non-zero vector $v^0 \in V$ and a distinguished non-zero covector $\varphi^0 \in V^*$ (or an affine hyperspace A) such that $\varphi^0(v^0) = 0$ (or $v^0 \in \mathsf{V}(A)$). Morphisms between bispecial vector spaces $\mathbf{V}_i = (V_i, v_i^0, \varphi_i^0)$, $i = 1, 2$, are those linear maps $F : V_1 \to V_2$ which respect the distinguished vectors and covectors: $F(v_1^0) = v_2^0$, $F^*(\varphi_2^0) = \varphi_1^0$.

In finite dimensions we have the obvious equivalence between the category of special affine spaces and the category of bispecial vector spaces. Since the category of bispecial vector spaces, which is canonically equivalent to the category of special affine spaces, is self-dual with the obvious duality $(V, v^0, \varphi^0)^\# = (V^*, \varphi^0, v^0)$, we have the natural duality $\mathbf{A} \leftrightarrow \mathbf{A}^\#$ in the category of special affine spaces. The dual $\mathbf{A}^\#$ of the special affine space $\mathbf{A} = (A, v^0)$ is thus the affine hyperspace in $(\hat{A})^* = A^\dagger$ defined as the level-1 set of the functional v^0 (we use the embedding $\hat{A} \subset \hat{A}^{**}$) and equipped with the vector 1_A, $1_A(A) = 1$, of its model vector space. In other words,

$$\mathbf{A}^\# = \mathbf{Aff}(\mathbf{A}, \mathbf{I}),$$

where $\mathbf{I} = (\mathbb{K}, 1)$ is the canonical special vector space, with canonically chosen map 1_A in $\mathsf{V}(\mathbf{Aff}(\mathbf{A}, \mathbf{I}))$. Let us observe that 1_A really belongs to the model vector space for $\mathbf{A}^\#$, since the latter consists of those affine maps $\varphi : A \to \mathbb{K}$ whose linear part vanishes on v^0, i.e.

$$\begin{aligned}\mathsf{V}(\mathbf{A}^\#) &= \{\varphi \in \mathsf{Aff}(A, \mathbb{K}) : \varphi_{\mathsf{V}}(v^0) = 0\} = \mathsf{Aff}(A/\langle v^0\rangle; \mathbb{K})\\ &= \{\hat{\varphi} \in \mathsf{Hom}(\hat{A}; \mathbb{K}) : \hat{\varphi}(v^0) = 0\} = \mathsf{Hom}(\hat{A}/\langle v^0\rangle; \mathbb{K}).\end{aligned}$$

The bispecial interpretation of the special affine duality yields immediately the canonical isomorphism $(\mathbf{A}^\#)^\# = \mathbf{A}$. Note also that one can view $\mathbf{I}$ as $\{*\}^\dagger$, where $\{*\}$ is a single-point affine space, and that the map $\varphi \mapsto -\varphi$ establishes a canonical isomorphism

$$\overline{\mathbf{A}^\#} \simeq \bar{\mathbf{A}}^\#.$$

A *special affine pairing* between special affine spaces $\mathbf{A}_1$ and $\mathbf{A}_2$ is a special biaffine map

$$\Phi : \mathbf{A}_1 \times \mathbf{A}_2 \to \mathbf{I}$$

for which the corresponding maps

$$\Phi^l : \mathbf{A}_1 \to \mathbf{A}_2^\# = \mathbf{Aff}(\mathbf{A}_2, \mathbf{I}), \qquad \Phi^l_{a_1}(a_2) = \Phi(a_1, a_2),$$

and

$$\Phi^r : \mathbf{A}_2 \to \mathbf{A}_1^\# = \mathbf{Aff}(\mathbf{A}_1, \mathbf{I}), \qquad \Phi^r_{a_2}(a_1) = \Phi(a_1, a_2)$$

are isomorphisms (in finite dimension it is sufficient that they are injective). An example is given by the canonical special affine pairing of dual special affine spaces

$$\langle\cdot,\cdot\rangle_{sa} : \mathbf{A} \times \mathbf{A}^\# \to \mathbf{I}, \qquad \langle a, \varphi\rangle_{sa} = \varphi(a) = a(\varphi)$$

This is just the restriction of the pairing between $\hat{\mathbf{A}}$ and $\hat{\mathbf{A}}^\# = \mathbf{A}^\dagger = \hat{\mathbf{A}}^*$ to the product of affine hyperspaces $\mathbf{A} \times \mathbf{A}^\#$. Note that every special affine map $\psi \in \mathbf{Aff}(\mathbf{A}_1; \mathbf{A}_2)$ has its dual $\psi^\# \in \mathbf{Aff}(\mathbf{A}_2^\#; \mathbf{A}_1^\#)$ defined by

$$\langle a_1, \psi^\#(a_2^\#)\rangle_{sa} = \langle \psi(a_1), a_2^\#\rangle_{sa}.$$

Note also that the concept of special vector spaces and the corresponding duality has been introduced in [24].

Since morphisms of bispecial vector spaces are linear maps which are simultaneously morphisms of special and cospecial structures, we can combine the constructions of the previous section to get products, direct sums, and tensor products in the category of special affine spaces.

Recall that the special direct sum $\mathbf{V}_1 \oplus^{sv} \mathbf{V}_2$ is represented by the quotient vector space $V_1 \oplus V_2/\langle v_1^0 - v_2^0\rangle$ with the distinguished vector being the class $[v_1^0]$ of v_1^0 (or, equivalently, the class $[v_2^0]$ of v_2^0). A similar construction $\mathbf{A}_1 \boxtimes \mathbf{A}_2 = ((A_1 \times A_2)/\langle(v_1^0, -v_2^0)\rangle, [(v_1^0, v_2^0)])$ we can perform in the category of special affine spaces. The model space for $\mathbf{A}_1 \boxtimes \mathbf{A}_2$, which will be called the *reduced product*, is canonically isomorphic with $\mathsf{V}(\mathbf{A}_1) \oplus^{sv} \mathsf{V}(\mathbf{A}_2)$. However, $\mathbf{A}_1 \boxtimes \mathbf{A}_2$ is not the direct sum in the category of special affine spaces which will be constructed in a while. The class of $u + v$ in $\mathsf{V}(\mathbf{A}_1) \oplus^{sv} \mathsf{V}(\mathbf{A}_2)$ (resp., the class of (a_1, a_2)

in $\mathbf{A}_1 \boxtimes \mathbf{A}_2$) we will denote by $u \oplus^{sv} v$ (resp., $a_1 \boxtimes a_2$). Note that any special affine pairing $\Phi : \mathbf{A}_1 \times \mathbf{A}_2 \to \mathbf{I}$ is constant on fibers of the canonical projection $\mathbf{A}_1 \times^a \mathbf{A}_2 \to \mathbf{A}_1 \boxtimes \mathbf{A}_2$. The notion of the reduced product is useful because of the following fact which can be easily derived from Theorem 4 (3).

Theorem 5. *For special affine spaces* $\mathbf{A}_i$, $i = 1, 2$, *we have*

$$(\mathbf{A}_1 \boxtimes \mathbf{A}_2)^{\#} \simeq \mathbf{A}_1^{\#} \boxtimes \mathbf{A}_2^{\#}.$$

In particular, for any affine space A *and* $\mathbf{A}_1 = A \times^a \mathbf{I}$, *one has* $A \times^a \mathbf{A}_2 = \mathbf{A}_1 \boxtimes \mathbf{A}_2$ *and consequently*

$$(A \overset{a}{\times} \mathbf{A}_2)^{\#} \simeq A^{\dagger} \boxtimes \mathbf{A}_2^{\#}.$$

For special affine spaces $\mathbf{A}_i = (A_i, v_i^0)$, $i = 1, 2$, their *special affine direct sum* $\mathbf{A}_1 \oplus^{sa} \mathbf{A}_2$ is represented by the affine space $(A_1 \oplus^a A_2)/\langle v_1^0 - v_2^0 \rangle$ modeled on $\mathrm{Ker}(1_{A_1} + 1_{A_2})/\langle v_1^0 - v_2^0 \rangle$ in $(\hat{A}_1 \oplus \hat{A}_2)/\langle v_1^0 - v_2^0 \rangle$ with the distinguished vector $v^0 \in (\hat{A}_1 \oplus \hat{A}_2)/\langle v_1^0 - v_2^0 \rangle$ being the class $[v_1^0]$ of v_1^0 (or, equivalently, the class $[v_2^0]$ of v_2^0). There are obvious special affine embeddings of $\mathbf{A}_i$ into $\mathbf{A}_1 \oplus^{sa} \mathbf{A}_2$, $i = 1, 2$.

The *special affine product* $\mathbf{A}_1 \times^{sa} \mathbf{A}_2$ is represented by $A_1 \times^a A_2$ modeled on $\mathsf{V}(\mathbf{A}_1) \times^{sv} \mathsf{V}(\mathbf{A}_2)$ with distinguished vector $v^0 = (v_1^0, v_2^0)$ in $\mathsf{V}(\mathbf{A}_1) \times \mathsf{V}(\mathbf{A}_2)$. The special affine projections from $\mathbf{A}_1 \times^{sa} \mathbf{A}_2$ onto $\mathbf{A}_i$, $i = 1, 2$, are obvious. Note that the dimensions of $\mathbf{A}_1 \oplus^{sa} \mathbf{A}_2$ and $\mathbf{A}_1 \times^{sa} \mathbf{A}_2$ are equal, but the model vector spaces are different (we have inclusion $\mathsf{V}(\mathbf{A}_1) \times \mathsf{V}(\mathbf{A}_2) \subset \mathrm{Ker}(1_{A_1} + 1_{A_2})$, but $v_1^0 - v_2^0 \in \mathsf{V}(\mathbf{A}_1) \times \mathsf{V}(\mathbf{A}_2)$). However, like for vector spaces, they are related by duality.

Theorem 6. *There are canonical isomorphisms*

$$(\mathbf{A}_1 \overset{sa}{\times} \mathbf{A}_2)^{\#} \simeq \mathbf{A}_1^{\#} \overset{sa}{\oplus} \mathbf{A}_2^{\#}, \tag{9}$$

$$(\mathbf{A}_1 \overset{sa}{\oplus} \mathbf{A}_2)^{\#} \simeq \mathbf{A}_1^{\#} \overset{sa}{\times} \mathbf{A}_2^{\#}. \tag{10}$$

For special multi-affine morphisms from $\mathbf{A}_1 \times \cdots \times \mathbf{A}_k$ we have a representing object, the *special affine tensor product* $\mathbf{A}_1 \otimes^{sa} \cdots \otimes^{sa} \mathbf{A}_k$, such that

$$\mathbf{Aff}^k(\mathbf{A}_1, \ldots, \mathbf{A}_k; \mathbf{A}) = \mathbf{Aff}(\mathbf{A}_1 \overset{sa}{\otimes} \cdots \overset{sa}{\otimes} \mathbf{A}_k; \mathbf{A}).$$

This is the quotient of the affine tensor product $A_1 \otimes^a \cdots \otimes^a A_k$ by the linear subspace of $\mathsf{V}(A_1 \otimes^a \cdots \otimes^a A_k)$ spanned by tensors

$$a_1 \overset{a}{\otimes} \cdots \overset{a}{\otimes} v_i^0 \overset{a}{\otimes} \cdots \overset{a}{\otimes} a_k - b_1 \overset{a}{\otimes} \cdots \overset{a}{\otimes} v_j^0 \overset{a}{\otimes} \cdots \overset{a}{\otimes} b_k,$$

where $a_l, b_l \in A_l$. In the special affine case $\dim(\mathbf{A}_1 \otimes^{sa} \mathbf{A}_2) = \dim(\mathbf{A}_1) \cdot \dim(\mathbf{A}_2)$. The canonical map

$$\mathbf{A}_1 \times \cdots \times \mathbf{A}_k \ni (a_1, \ldots, a_k) \mapsto a_1 \overset{sa}{\otimes} \cdots \overset{sa}{\otimes} a_k \in \mathbf{A}_1 \overset{sa}{\otimes} \cdots \overset{sa}{\otimes} \mathbf{A}_k,$$

where $a_1\otimes^{sa}\cdots\otimes^{sa}a_k$ is the coset of $a_1\otimes^{a}\cdots\otimes^{a}a_k$, is a special multi-affine map. We have obvious canonical isomorphisms

$$\mathbf{A}_1\overset{sa}{\otimes}\mathbf{A}_2\simeq\mathbf{A}_2\overset{sa}{\otimes}\mathbf{A}_1,\qquad \mathbf{A}_1\overset{sa}{\otimes}(\mathbf{A}_2\overset{sa}{\otimes}\mathbf{A}_3)\simeq(\mathbf{A}_1\overset{sa}{\otimes}\mathbf{A}_2)\overset{sa}{\otimes}\mathbf{A}_3,\qquad (\mathbf{A}_1\overset{sa}{\otimes}\mathbf{A}_2)^{\#}=\mathbf{A}_1^{\#}\overset{sa}{\otimes}\mathbf{A}_2^{\#}.$$

Note that there are no special k-affine and skew-symmetric maps $F:\mathbf{A}\times\cdots\times\mathbf{A}\to\mathbf{A}'$, $\mathbf{A}=(A,v^0)$, $\mathbf{A}'=(A',v')$ for $k>1$, since special k-affine implies that $F^1_{\mathsf{V}}(v^0,a,\dots)=F^2_{\mathsf{V}}(a,v^0,\dots)=v'$ and skew-symmetry that $F^1_{\mathsf{V}}(v^0,a,\dots)=-F^2_{\mathsf{V}}(a,v^0,\dots)$, so $v'=0$; a contradiction.

Starting with an object in vector or affine category we can always construct canonically an object in the special category just by taking the product with $\mathbf{I}=(\mathbb{R},1)$. For example, given an affine space A we can define its *specialization* $\mathbf{S}_A$ as the special affine space $\mathbf{S}_A=(A\times^a\mathbf{I},(0,1))$ modeled on the specialization $\mathbf{S}_{\mathsf{V}(A)}=(\mathsf{V}(A)\times\mathbf{I},(0,1))$ of the model space for A. Using the specialization we can describe certain canonical constructions in affine category in the language of the special affine category. Note that in this language $\mathbf{I}=\mathbf{S}_{\{*\}}$, where $\{*\}$ is a one-point affine space.

Theorem 7. *For affine spaces A, A_1, A_2 there are canonical isomorphisms*

(a) $\mathbf{S}^{\#}_A\simeq A^{\dagger}$, (11)

(b) $\hat{\mathbf{S}}_A\simeq\mathbf{S}_{\hat{A}}$, (12)

(c) $\mathbf{S}_{A_1\oplus^a A_2}\simeq\mathbf{S}_{A_1}\overset{sa}{\oplus}\mathbf{S}_{A_2}$, (13)

(d) $\mathbf{S}_{A_1\times^a A_2}\simeq\mathbf{S}_{A_1}\boxtimes\mathbf{S}_{A_2}$. (14)

Proof. The proof is straightforward and we will prove only (a) leaving the rest to the reader. Since any $\varphi\in\mathbf{S}^{\#}_A=\mathbf{Aff}(\mathbf{S}_A;\mathbf{I})$ is an affine map characterized by $\varphi(a,r)=\varphi(a,0)+r$, there is a one–one correspondence

$$\mathbf{S}^{\#}_A\ni\varphi\mapsto\varphi^{\dagger}\in A^{\dagger}$$

given by $\varphi^{\dagger}(a)=\varphi(a,0)$. It is obvious that $(\varphi+1_{\mathbf{S}_A})^{\dagger}=\varphi^{\dagger}+1_A$, so this is an isomorphism of special affine spaces. □

5. Spaces of affine scalars

In this section we will consider one-dimensional special affine spaces $\mathbf{Z}=(Z,v^0)$. Since the model vector space $\mathsf{V}(\mathbf{Z})$ is one-dimensional and special, we can identify it with $\mathbf{I}=(\mathbb{K},1)$. In what follows we assume that $\mathbf{Z}$ is modeled on $\mathbf{I}$. In this picture the adjoint special affine space $\bar{\mathbf{Z}}$ is isomorphic to $(\bar{Z}^a,1)$, i.e. $\bar{\mathbf{Z}}$ is Z with the same distinguished vector but with the adjoint affine structure: $\sigma-_0\sigma'=\sigma'-\sigma$ (or $\sigma+_0 r=\sigma+(-r)$). The points of $\mathbf{Z}$ are like numbers, i.e. elements of $\mathbb{K}$, but the origin 0 is not fixed, so only the difference of points makes sense as a number or, equivalently, we can add numbers to points of $\mathbf{Z}$. We will call $\mathbf{Z}$ a *space of affine scalars*. Of course, any point $\sigma_0\in\mathbf{Z}$ defines the isomorphism $I_{\sigma_0}:\mathbf{Z}\to\mathbf{I}$, $\sigma\mapsto\sigma-\sigma_0$, of special affine bundles. We can consider also the map $\mathbf{F}:Z\to Z^{\dagger}=\mathsf{Aff}(Z,\mathbb{K})$ given by $\mathbf{F}_\sigma(\sigma')=\sigma-\sigma'$. The following is straightforward.

Theorem 8. *The map* $\sigma \mapsto \mathbf{F}_\sigma$ *induces a canonical isomorphism* $\mathbf{F} : \mathbf{Z} \to \bar{\mathbf{Z}}^\#$ *represented by the special affine pairing*

$$\mathbf{Z} \times \bar{\mathbf{Z}} \ni (\sigma, \sigma') = \mathbf{F}_\sigma(\sigma') = \sigma - \sigma' \in \mathbf{I}.$$

This isomorphism extends by linearity to an isomorphism $\mathbf{F} : \hat{\mathbf{Z}} \to \mathbf{Z}^\dagger$ *of special vector spaces*.

There are canonical geometric structures on the space of affine numbers $\mathbf{Z}$. Since a translation of a polynomial function on $\mathbb{K}$ is a polynomial function, the algebra $\mathrm{Pol}(\mathbf{Z})$ of polynomial functions on $\mathbf{Z}$ is well-defined. It is generated by affine functions on $\mathbf{Z}$. There is a canonical 'vector field' (derivation of $\mathrm{Pol}(\mathbf{Z})$) on $\mathbf{Z}$ being the 'fundamental vector field' $X_\mathbf{Z}$ of the $\mathbb{K}$-action on $\mathbf{Z}$, $\sigma \mapsto \sigma + s$. With respect to any 'global coordinate system' $I_\sigma : \mathbf{Z} \to \mathbb{K}$ this vector field has the form $X_\mathbf{Z} = -\partial_s$, where $\partial_s(s^n) = ns^{n-1}$ for s being the standard coordinate in $\mathbb{K}$. We can also consider a Jacobi structure on $\mathbf{Z}$ with the corresponding Jacobi bracket

$$\{f, g\}_\mathbf{Z} = fX_\mathbf{Z}(g) - gX_\mathbf{Z}(f) \tag{15}$$

on $\mathrm{Pol}(\mathbf{Z})$. Of course, in the case $\mathbb{K} = \mathbb{R}$ one can understand $X_\mathbf{Z}$ as a true vector field on $\mathbf{Z}$ and the bracket $\{\cdot, \cdot\}_\mathbf{Z}$ can be understood as a bracket defined on the algebra $C^\infty(\mathbf{Z})$ of all smooth functions on $\mathbf{Z}$.

Proposition 1.

(a) *For all* $\sigma, \sigma' \in \mathbf{Z}$:

$$\{\mathbf{F}_\sigma, \mathbf{F}_{\sigma'}\}_\mathbf{Z} = \mathbf{F}_\sigma(\sigma') = \sigma - \sigma'; \tag{16}$$

(b) *For all* $\phi \in \mathbf{Z}^\dagger$ *and all* $\sigma \in \mathbf{Z}$:

$$\{\phi, \mathbf{F}_\sigma\}_\mathbf{Z} = \phi(\sigma). \tag{17}$$

Proof. Let us identify $\mathbf{Z}$ with $\mathbf{I}$ by fixing certain $\sigma_0 \in \mathbf{Z}$ and let s be the linear coordinate on $\mathbf{I}$. Then, $\mathbf{F}_\sigma(s) = \sigma - s$ and, for $\phi(s) = as + b$, we have

$$\{as + b, \sigma - s\}_\mathbf{Z} = -(as + b)\partial_s(\sigma - s) + (\sigma - s)\partial_s(as + b) = (a\sigma + b) = \phi(\sigma)$$

that proves (a). Part (b) follows from (a) easily. □

Note that the vector space $\mathbf{Z}^\dagger \simeq \hat{\mathbf{Z}}$ is two-dimensional but there is no canonical basis. Instead, we have the canonical exact sequence

$$0 \to \mathbf{I} \to \mathbf{Z}^\dagger \to \mathbf{I} \to 0,$$

where the inclusion is $\mathbf{I} \ni \lambda \mapsto \lambda 1_\mathbf{Z}$ and the projection $\mathbf{Z}^\dagger \ni \phi \mapsto -X_\mathbf{Z}(\phi) \in \mathbf{I}$ gives the 'directional coefficient' of affine functions. The affine subspace $\mathbf{Z}^\#$ in $\mathbf{Z}^\dagger$ is characterized as the family of affine functions ϕ on $\mathbf{Z}$ for which $X_\mathbf{Z}(\phi) = -1$. Similarly, the image of $\mathbf{Z}$

under the isomorphism $\mathbf{F} : \hat{\mathbf{Z}} \to \mathbf{Z}^\dagger$ is characterized by $X_{\mathbf{Z}}(\phi) = 1$. The Jacobi bracket (15) describes the pairing between $\mathbf{Z}^\dagger$ and $\hat{\mathbf{Z}}$.

Theorem 9. *For all $\phi \in \mathbf{Z}^\dagger$ and all $u \in \hat{\mathbf{Z}}$:*

$$\{\phi, \mathbf{F}_u\}_{\mathbf{Z}} = \langle \phi, u \rangle. \tag{18}$$

Proof. The theorem follows easily from (17) by linearity. □

Remark. We cannot add two affine scalars. However, for spaces $\mathbf{Z}$, $\mathbf{Z}'$ of affine scalars we can introduce an equivalence relation in $\mathbf{Z} \times \mathbf{Z}'$ by

$$(z, z') \sim (z_1, z'_1) \Leftrightarrow z - z_1 = z'_1 - z'$$

and interpret the equivalence class of (z, z') as a sum of z and z'. We recognize the space of such equivalence classes as $\mathbf{Z} \boxtimes \mathbf{Z}'$. Let us remark that this concept of addition of affine scalars is already present in [23].

6. Affine and special affine bundles

All above can be formulated *mutatis mutandis* for affine bundles instead of affine spaces. Here $\mathbb{K} = \mathbb{R}$ and affine bundles are smooth bundles of affine spaces which are locally trivial, so that we pass from one local trivialization to another using the group of affine transformations. Since we do everything fiberwise over the same base manifold M and consider only morphisms over the identity map on the base (if not explicitly stated otherwise), this generalization is straightforward and we use, in principle, the same notation. For instance, $\mathsf{V}(A)$ denotes the vector bundle which is the model for an affine bundle $\zeta : A \to M$ over a base manifold M. By Sec we denote the spaces of sections, e.g. $\mathsf{Sec}(\zeta)$ (or sometimes $\mathsf{Sec}(A)$) is the affine space of sections of the affine bundle $\zeta : A \to M$. This time, however, we must distinguish the bundles of morphisms like $\mathsf{Aff}_M(A_1, A_2)$, $\mathsf{Hom}_M(V_1, V_2)$, etc., from their spaces of sections which consist of particular morphisms. We will write shortly $\mathsf{Aff}(A_1, A_2)$ instead of $\mathsf{Sec}(\mathsf{Aff}_M(A_1, A_2))$, etc., and $A^\dagger = \mathsf{Aff}_M(A, \mathbb{R})$ (resp., $V^* = \mathsf{Hom}_M(V, \mathbb{R})$) instead of $\mathsf{Aff}_M(A, M \times \mathbb{R})$ (resp., $\mathsf{Hom}_M(V, M \times \mathbb{R})$) and $\mathsf{Aff}(A)$ (resp., $\mathsf{Lin}(V)$) for the space of sections—affine functions on A (resp., linear functions on V).

Every section v of the model vector bundle $\mathsf{V}(A)$ induces a vertical vector field v_A on A (called the *vertical lift* of V) being the generator of the one parameter group of translations $A \ni \sigma_m \mapsto \sigma_m + sv(m)$. Of course, v is uniquely determined by v_A. By a *special vector bundle* we understand, clearly, a vector bundle with a distinguished nowhere vanishing section. Consequently, a *special affine bundle* is an affine bundle modeled on a special vector bundle, etc. Every special affine bundle $\mathbf{A} = (A, v^0)$ carries a distinguished vertical vector field $X_{\mathbf{A}} = -v^0_A$, being the fundamental vector field of the $(\mathbb{R}, +)$-action on A induced by v^0, i.e. the action. $A \ni \sigma_m \mapsto \sigma_m + sv^0(m)$, and thus a canonical Jacobi structure determined by $X_{\mathbf{A}}$. The corresponding Jacobi bracket of smooth functions on A reads

$$\{f, g\}_{\mathbf{A}} = fX_{\mathbf{A}}(g) - gX_{\mathbf{A}}(f).$$

If V is a vector subbundle in the model vector bundle $\mathsf{V}(A)$ of an affine bundle A over M, then the canonical projection $\rho : A \to A/V$ of A onto the quotient affine bundle A/V defines an affine bundle structure on the total space A over A/V modeled on $(A/V) \times_M V$ (see [2]). We will call this affine bundle an *affine projection bundle* (AP-bundle) and denote it $\mathsf{AP}(A, V)$. Since ρ is a morphism of affine bundles over M, it makes sense to speak about the *affine section bundle* $\mathsf{AS}(A, V)$ of ρ. The affine section bundle with fibers

$$\mathsf{AS}(A, V)_m = \{z_m \in \mathsf{Aff}(A_m/V_m; A_m) : z_m \circ \rho_m = \mathrm{id}_{A_m/V_m}\}$$

is an affine bundle over M modeled on $\mathsf{Aff}_M(A/V; V)$. The space of sections of the affine bundle $\mathsf{AS}(A, V)$, i.e. the space of affine sections of $\mathsf{AP}(A, V)$, we will denote by $\mathsf{Aff\,Sec}(A, V)$.

If, by chance, A is a vector bundle, then we can also speak about the *linear section bundle* $\mathsf{LS}(A, V)$ over M with fibers

$$\mathsf{LS}(A, V)_m = \{u_m \in \mathsf{Hom}(A_m/V_m; A_m) : u_m \circ \rho_m = \mathrm{id}_{A_m/V_m}\}.$$

This is an affine bundle over M modeled on $\mathsf{Hom}_M(A/V; V)$. The space of sections of the affine bundle $\mathsf{LS}(A, V)$, i.e. the space of linear sections of $\mathsf{AP}(A, V)$, will be denoted by $\mathsf{Lin\,Sec}(A, V)$.

Using the canonical extensions of affine maps from an affine space to linear maps from its vector hull we get the following variant of Theorem 2.

Theorem 10. *The canonical embedding* $\mathsf{Aff}_M(A/V; A) \subset \mathsf{Hom}_M(\hat{A}/V; \hat{A})$ *induces a canonical identification*

$$\mathsf{AS}(A, V) \simeq \mathsf{LS}(\hat{A}, V).$$

On the level of sections we denote this identification

$$\mathsf{Aff\,Sec}(A, V) \ni \sigma \mapsto \hat{\sigma} \in \mathsf{Lin\,Sec}(\hat{A}, V).$$

7. Bundles of affine values

A particularly interesting case is that for one-dimensional special affine bundles $\mathbf{Z} = (Z, v^0)$ over M which we will call *bundles of affine values* (AV-bundles) and usually denote by $\mathbf{Z}$. The fibers of such bundles are spaces of affine scalars described in Section 5. The sections of an AV-bundle will play the role of functions in our affine differential geometry that will be developed in next sections. The model vector bundle $\mathsf{V}(\mathbf{Z})$ for $\xi : \mathbf{Z} \to M$ is one-dimensional and equipped with a distinguished non-vanishing section. It is clear that this yields a canonical identification of $\mathsf{V}(\mathbf{Z})$ with the trivial bundle $M \times \mathbb{R}$ with distinguished non-vanishing section represented by the constant function 1_M, i.e. with $M \times \mathbf{I}$. Thus the AV-bundle $\mathbf{Z}$ itself is *trivializable*, since every section $\sigma \in \mathsf{Sec}(\mathbf{Z})$ defines the isomorphism

$I_\sigma : \mathbf{Z} \to \mathsf{V}(\mathbf{Z}) = M \times \mathbf{I}$, but *not trivial*, because we have no canonical trivialization. We insist on not introducing any particular trivialization, since introducing it is like fixing a frame or observer in a physical system and our approach is thought of as a geometric framework for studying such systems in a frame-independent way.

The sections of $\mathbf{Z}$ can be viewed as 'functions with affine values', since they take values in fibers of $\mathbf{Z}$ which are almost reals except for the fact that we do not know where is 0, so we can only measure the relative positions of points. The main difference and difficulty is now that $\mathsf{Sec}(\mathbf{Z})$ is not an algebra nor even a vector space but only an affine space modeled on the algebra $C^\infty(M)$ of smooth functions. In what follows, we will identify the model bundle for an AV-bundle $\mathbf{Z}$ with $M \times \mathbf{I}$. Thus we can add reals, $z_m \mapsto z_m + s$, in every fiber Z_m of Z, so we have a free and transitive on fibers action of the group $(\mathbb{R}, +)$ on $\mathbf{Z}$, i.e. $\mathbf{Z}$ is an $\mathbb{R}$-principal bundle. Let us recall that the vertical vector field on $\mathbf{Z}$ which is the fundamental vector field of this action we denote by $X_{\mathbf{Z}}$ and the corresponding vertical Jacobi bracket on $\mathbf{Z}$ by $\{\cdot, \cdot\}_{\mathbf{Z}}$. The adjoint special affine bundle $\bar{\mathbf{Z}}$ is represented by Z with the opposite action of $\mathbb{R}$, i.e. with the fundamental vector field $-X_{\mathbf{Z}}$. Conversely, it is easy to see that every $\mathbb{R}$-principal bundle Z carries an AV-bundle structure. We have an obvious bundle version of Theorem 8.

Theorem 11. *There is a canonical isomorphism*

$$\mathbf{F} : \mathbf{Z} \to \bar{\mathbf{Z}}^\#, \qquad \mathbf{F}_{a_m}(a'_m) = a_m - a'_m, \tag{19}$$

represented by the the special affine pairing

$$\mathbf{Z} \times \bar{\mathbf{Z}} \ni (a_m, a'_m) \mapsto a_m - a'_m \in \mathbf{I}.$$

This isomorphism extends by linearity to an isomorphism $\mathbf{F} : \hat{\mathbf{Z}} \to \mathbf{Z}^\dagger$ *of special vector bundles.*

$\mathbf{F} : \hat{\mathbf{Z}} \to \mathbf{Z}^\dagger$ defines also a map on the level of sections, $u \in \mathsf{Sec}(\hat{\mathbf{Z}}) \mapsto \mathbf{F}_u \in \mathsf{Aff}(\mathbf{Z})$. Since $M \times \mathbf{I} \hookrightarrow \hat{\mathbf{Z}}$ as $\mathsf{V}(\mathbf{Z})$, we can understand 1_M as a section of $\hat{\mathbf{Z}}$ and we obtain $\mathbf{F}_{1_M} = 1_{\mathbf{Z}}$, so the map $\mathbf{F}$ identifies functions on M with their pull-backs to $\mathbf{Z}$. Moreover, for any $\sigma \in \mathsf{Sec}(\mathbf{Z})$, the function $\mathbf{F}_\sigma$ is an affine function on $\mathbf{Z}$ which is uniquely characterized by the property that $\mathbf{F}_\sigma$ vanishes on the image of $\sigma \in \mathsf{Sec}(\mathbf{Z})$ and $X_{\mathbf{Z}}(\mathbf{F}_\sigma) = 1$. This allows us to understand sections of $\mathbf{Z}$ as smooth functions φ on $\mathbf{Z}$ with $X_{\mathbf{Z}}(\varphi) = 1$. The space of sections of $\bar{\mathbf{Z}}$ is identified with the space of smooth functions on $\mathbf{Z}$ satisfying $X_{\mathbf{Z}}(\varphi) = -1$.

An important observation is that every special affine bundle $\mathbf{A} = (A, v^0)$ gives rise to an AV-bundle. Indeed, the vector bundle $\mathsf{Aff}_M(A/\langle v^0\rangle; \langle v^0\rangle)$ is special. As the distinguished section $\tilde{v}^0$, which is constant on fibers of $A/\langle v^0\rangle$ we chose $\tilde{v}^0(\rho(a_m)) = -v^0(m)$. Hence, $\mathsf{AP}(A, \langle v^0\rangle)$ is canonically an AV-bundle which we denote by $\mathsf{AV}(\mathbf{A})$. The distinguished section is chosen in such a way that $X_{\mathsf{AV}(\mathbf{A})}$ is the vertical lift $v^0_{\mathbf{A}}$ of v^0, so $\mathsf{AV}(\mathbf{Z}) = \bar{\mathbf{Z}}$ and $\mathsf{AV}(\mathbf{Z}^\#) = \mathbf{Z}$ for any AV-bundle $\mathbf{Z}$. Moreover the map $\mathbf{F}$ for $\mathsf{AV}(\mathbf{A})$ is characterized by the property that the affine function $\mathbf{F}_\sigma$ associated with a section σ of $\mathsf{AV}(\mathbf{A})$ satisfies $v^0_{\mathbf{A}}(\mathbf{F}_\sigma) = 1$ and $\mathbf{F}_\sigma \circ \sigma = 0$. Note the isomorphism $\overline{\mathsf{AV}(\mathbf{A})} = \mathsf{AV}(\bar{\mathbf{A}})$. The choice of the distinguished section in $\mathsf{AV}(\mathbf{A})$ is justified by the next two theorems.

In the linear case there is an obvious identification of sections X of a vector bundle E with linear functions $\iota_{E^*}(X)$ on the dual bundle E^*, defined by the canonical pairing. If E' is a submanifold of E^* (in applications E' will be usually a vector or an affine subbundle), the restriction of $\iota_{E^*}(X)$ to E' will be denoted by $\iota_{E'}(X)$. In this notation, a section a of a special affine bundle $\mathbf{A}$ (regarded as a section of $\hat{\mathbf{A}}$) will give rise to a linear function $\iota_{\mathbf{A}^\dagger}(a)$ on $\mathbf{A}^\dagger$ and an affine function $\iota_{\mathbf{A}^\#}(a)$ on the affine subbundle $\mathbf{A}^\#$ of $\mathbf{A}^\dagger$. Denote the map $\mathbf{F}$ for the AV-bundle $\mathsf{AV}(\mathbf{A}^\dagger)$ (resp., $\mathsf{AV}(\mathbf{A}^\#)$) by $\mathbf{F}^\dagger$ (resp., $\mathbf{F}^\#$).

For a special affine bundle (resp., a special vector bundle) $\mathbf{A} = (A, v^0)$ denote $\mathsf{AS}(A, \langle v^0\rangle)$ by $\mathsf{AS}(\mathbf{A})$ (resp., $\mathsf{LS}(A, \langle v^0\rangle)$ by $\mathsf{LS}(\mathbf{A})$). The spaces of sections of these bundles we denote simply $\mathsf{Aff\,Sec}(\mathbf{A})$ and $\mathsf{Lin\,Sec}(\mathbf{A})$, respectively. Since the section $\tilde{v}^0$ is affine, also $\mathsf{AS}(\mathbf{A})$ is canonically a special affine bundle. In the case when $\mathbf{A}$ is a special vector bundle the affine bundle $\mathsf{LS}(\mathbf{A})$ is not canonically special, since the section $\tilde{v}^0$ is not linear. However, in the case when $\mathbf{A}$ is a bispecial vector bundle with the distinguished section φ^0 of $\mathbf{A}^*$, $\langle \varphi^0, v^0\rangle = 0$, then also $\mathsf{LS}(\mathbf{A})$ is special affine with the distinguished section $\widetilde{v_{\varphi^0}} \in \mathsf{Hom}(A/\langle v^0\rangle; \langle v^0\rangle)$,

$$\widetilde{v_{\varphi^0}}(\rho(a_m)) = -\langle \varphi^0(m), a_m\rangle v^0(m).$$

Theorem 12 (Grabowska et al. [2]). *There is a canonical isomorphism of affine bundles*

$$A \simeq \mathsf{LS}(A^\dagger), \qquad a_m \mapsto \hat{\sigma}_{a_m},$$

where

$$\hat{\sigma}_{a_m}([\varphi_m]) = \varphi_m - \varphi_m(a_m)1_A(m).$$

In other words, for any section a of A,

$$\mathbf{F}^\dagger_{\hat{\sigma}_a} = \iota_{A^\dagger}(a).$$

The corresponding isomorphism of the model vector bundles takes the form

$$\mathsf{V}(A)_m \ni X_m \leftrightarrow -\iota^\dagger_{X_m} \in (A^\dagger/\langle 1_A\rangle)^*,$$

where $\iota^\dagger_{X_m}([\varphi_m]) = (\varphi_m)_{\mathsf{v}}(X_m)$.

Note that the above theorem is an affine version of the well-known fact that sections of a vector bundle E over M can be identified with linear (along fibers) functions on the dual E^*, i.e. with linear sections of the bundle $E^* \times \mathbb{R}$ over E^*. We can extend this identification to special affine bundles as follows.

Theorem 13. *For a special affine bundle $\mathbf{A} = (A, v^0)$ there is a canonical identification of special affine bundles*

$$\mathbf{A} \simeq \mathsf{AS}(\mathbf{A}^\#) \simeq \mathsf{LS}(\mathbf{A}^\dagger), \qquad a_m \leftrightarrow \sigma_{a_m} \leftrightarrow \hat{\sigma}_{a_m}. \tag{20}$$

On the level of sections it takes the form

$$\mathsf{Sec}(\mathbf{A}) \simeq \mathsf{Aff\,Sec}(\mathbf{A}^\#) \simeq \mathsf{Lin\,Sec}(\mathbf{A}^\dagger), \qquad a \leftrightarrow \sigma_a \leftrightarrow \hat{\sigma}_a, \tag{21}$$

where

$$\mathbf{F}^{\dagger}_{\hat{\sigma}_a} = \iota_{\mathbf{A}^{\dagger}}(a),$$

$$\mathbf{F}^{\#}_{\sigma_a} = \iota_{\mathbf{A}^{\#}}(a).$$

This identification leads to the obvious identification of the corresponding model vector bundles

$$\mathsf{V}(\mathbf{A}) = \mathsf{Aff}_M(\mathbf{A}^{\#}/\langle 1_{\mathbf{A}}\rangle;\mathbb{R}) = \mathsf{Hom}_M(\mathbf{A}^{\dagger}/\langle 1_{\mathbf{A}}\rangle;\mathbb{R})(= (\mathbf{A}^{\dagger}/\langle 1_{\mathbf{A}}\rangle)^*)$$

taking on sections the form

$$X \leftrightarrow -\iota^{\#}_X \leftrightarrow -\iota^{\dagger}_X,$$

where linear functions $\iota^{\dagger}_X$ and affine functions $\iota^{\#}_X$ on $\mathbf{A}^{\dagger}/\langle 1_{\mathbf{A}}\rangle$ and $\mathbf{A}^{\#}/\langle 1_{\mathbf{A}}\rangle$ are the projections of linear functions $\iota_{\mathbf{A}^{\dagger}}(X)$ on $\mathbf{A}^{\dagger}$ and affine functions $\iota_{\mathbf{A}^{\#}}(X)$ on $\mathbf{A}^{\#}$, respectively.

Proof. The proof that these bundles are canonically isomorphic is just the combination of Theorems 12 and 10. That the distinguished sections are preserved follows from

$$\mathbf{F}^{\dagger}_{\hat{\sigma}_{a+v^0}} = \iota_{\mathbf{A}^{\dagger}}(a+v^0) = \mathbf{F}^{\dagger}_{\hat{\sigma}_a} + \iota_{\mathbf{A}^{\dagger}}(v^0), \qquad \mathbf{F}^{\#}_{\sigma_{a+v^0}} = \iota_{\mathbf{A}^{\#}}(a+v^0) = \mathbf{F}^{\#}_{\sigma_a} + 1. \qquad \square$$

Corollary 1. *For an affine bundle A and an AV-bundle $\mathbf{Z}$ over M there is a canonical identification*

$$\mathsf{Aff}_M(A;\mathbf{Z}) \simeq \mathbf{A}^{\dagger} \boxtimes_M \mathbf{Z}.$$

Proof. Observe first that $A \times^a{}_M \mathbf{Z}$ is canonically a special affine bundle and the identification *mapping* $\leftrightarrow$ *graph* induces the identification of $\mathsf{Aff}_M(A;\mathbf{Z})$ with the space $\mathsf{AS}(A\times^a{}_M\bar{\mathbf{Z}})$ of affine sections of the associated AP-bundle $A\times^a{}_M\bar{\mathbf{Z}}$ over A. The latter is, due to the above theorem, canonically identified with the special affine bundle $(A\times^a{}_M\bar{\mathbf{Z}})^{\#}$ over M. In view of Theorems 5 and 11

$$\left(A \overset{a}{\times}_M \bar{\mathbf{Z}}\right)^{\#} \simeq \mathbf{A}^{\dagger} \boxtimes_M \bar{\mathbf{Z}}^{\#} \simeq \mathbf{A}^{\dagger} \boxtimes_M \mathbf{Z}. \qquad \square$$

We will end up this section with presenting the above concepts in local coordinates. First of all, for a special vector bundle $\mathsf{v}(\eta) : \mathbf{V} = (V, v^0) \to M$ we choose a coordinate neighborhood U in M with coordinates $x = (x^b)$ and a basis $(v^1, \dots, v^k, v^0)$ of local sections over U which contains the distinguished v^0. On fibers over U we have then the associated linear coordinates $(y, s) = (y_1, \dots, y_k, s)$, so the coordinates (x, y, s) on $(\mathsf{v}(\eta))^{-1}(U)$. We will call such local coordinates on $\mathbf{V}$ *linear coordinates*. These coordinates can serve as coordinates on $\eta^{-1}(U)$ for any special affine bundle $\eta : \mathbf{A} = (A, v^0) \to M$ modeled on $\mathsf{v}(\eta)$ if we use the isomorphism of affine bundles $I_\sigma : A \to V = \mathsf{V}(A)$ determined by a section $\sigma \in \mathsf{Sec}(A)$. Such coordinates will be called *local affine coordinates on* $\mathbf{A}$. The change of the section σ results in the transformation of coordinates by a translation $(x, y, s) \mapsto (x, y + f(x), s + g(x))$, so that objects of affine differential geometry should be defined in local coordinates invariantly with respect to this change of coordinates. On $\hat{\mathbf{A}}$ we have linear coordinates (x, y, z, s) such that A is characterized by the equation $z = 1$. The canonical vector field on $\mathbf{A}$ has the expression $X_{\mathbf{A}} = -\partial_s$. Affine functions on A have the form

$$\varphi(x, y, s) = \alpha^i(x)y_i + \gamma(x)s + \beta(x)$$

and correspond to linear functions

$$\hat{\varphi}(x, y, z, s) = \alpha^i(x)y_i + \gamma(x)s + \beta(x)z$$

on $\hat{\mathbf{A}}$. Hence, $(x^b, \alpha^1, \dots, \alpha^k, \beta, \gamma)$ represent coordinates on $\mathbf{A}^\dagger$. The distinguished section is $1_{\mathbf{A}}(x) = (x, 0, 1, 0)$. The affine subspace $\mathbf{A}^\#$ in $\mathbf{A}^\dagger$ is characterized by $\gamma = 1$.

If $\mathbf{A} = \mathbf{Z}$ is an AV-bundle then the coordinates y are lacking and the affine function corresponding to the section $\sigma : s = \sigma(x)$ is $\mathbf{F}_\sigma(x, s) = \sigma(x) - s$. For the particular case of the AV-bundles $\mathsf{AV}(\mathbf{A}^\#)$ and $\mathsf{AV}(\mathbf{A}^\dagger)$ induced by a special affine bundle $\mathbf{A}$ we have coordinate expressions $(x, \alpha, \beta) \mapsto (x, \alpha)$ and $(x, \alpha, \beta, \gamma) \mapsto (x, \alpha, \gamma)$, respectively. The distinguished sections are described by the equation $\beta = -1$. The canonical pairing between $\hat{\mathbf{A}}$ and $\mathbf{A}^\dagger$ is

$$\langle (x, y, z, s), (x, \alpha, \beta, \gamma)\rangle = y_i\alpha^i + z\beta + s\gamma,$$

so that the canonical pairing between $\mathbf{A}$ and $\mathbf{A}^\#$ reads

$$\langle (x, y, s), (x, \alpha, \beta)\rangle_{sa} = \langle (x, y, 1, s), (x, \alpha, \beta, 1)\rangle = y_i\alpha^i + \beta + s.$$

In other words, $\iota_{\mathbf{A}^\#}(a)(x, \alpha, \beta) = y_i(x)\alpha^i + \beta + s(x)$ for a section $a(x) = (x, y(x), s(x))$ of $\mathbf{A}$.

Affine (resp., linear) sections of the bundles $\mathsf{AV}(\mathbf{A}^\#)$ and $\mathsf{AV}(\mathbf{A}^\dagger)$ have the form

$$\sigma(x, \alpha) = (x, \alpha, y_i(x)\alpha^i + s(x)) \quad \text{and} \quad \hat{\sigma}(x, \alpha, \gamma) = (x, \alpha, \gamma, y_i(x)\alpha^i(x) + s(x)\gamma),$$

respectively. The associated affine function $\mathbf{F}^\#_\sigma = i_{\mathbf{A}^\#}(\bar{a})$ on $\mathbf{A}^\#$ reads

$$\mathbf{F}_\sigma(x, \alpha, \beta) = \beta - y_i(x)\alpha^i - s(x) = \langle (x, \alpha, \beta), (x, -y(x), -s(x))\rangle_{sa}$$

and corresponds to the section $\bar{a}(x) = (x, -y(x), -s(x))$ of $\mathbf{A}$. Conversely, the section $a(x)$ corresponds, by definition, to the affine section

$$\sigma_a(x, \alpha) = (x, \alpha, \beta - \langle (x, y(x), s(x)), (x, \alpha, \beta)\rangle_{sa}) = (x, \alpha, -y_i(x)\alpha^i - s(x))$$

of $\mathbf{A}^\#$.

8. AV-differential geometry: the phase and the contact bundles

The standard Cartan calculus of differential forms is based on the algebra of differentiable functions on a manifold M. We will start to build *AV-differential geometry* where for the Cartan calculus we replace functions by sections of an AV-bundle $\zeta : \mathbf{Z} \to M$ modeled on the trivial bundle $pr_M : M \times \mathbf{I}$. This is our starting object whose sections $\mathsf{Sec}(\mathbf{Z})$ replace the sections of $M \times \mathbb{R}$, i.e. smooth functions $C^\infty(M)$ on M in the standard differential geometry. This chapter is based on [22,25], where AV-analogs of the cotangent and contact bundles T^*M and $\mathsf{T}^*M \times \mathbb{R}$ have been introduced.

One builds an AV-analog of the cotangent bundle T^*M as follows. Let us define an equivalence relation in the set of all pairs (m, σ), where m is a point in M and σ is a section of ζ. Two pairs (m, σ) and (m', σ') are equivalent if $m' = m$ and $\mathrm{d}(\sigma' - \sigma)(m) = 0$. We have identified the section $\sigma' - \sigma$ of pr_M with a function on M for the purpose of evaluating the differential $\mathrm{d}(\sigma' - \sigma)(m)$. We denote by $\mathsf{P}\mathbf{Z}$ the set of equivalence classes. The class of (m, σ) will be denoted by $\mathbf{d}\sigma(m)$ or by $\mathbf{d}_m\sigma$ and will be called the *differential* of σ at m. We will write $\mathbf{d}$ for the affine exterior differential to distinguish it from the standard d. We define a mapping $\mathsf{P}\zeta : \mathsf{P}\mathbf{Z} \to M$ by $\mathsf{P}\zeta(\mathbf{d}\sigma(m)) = m$. The bundle $\mathsf{P}\zeta$ is canonically an affine bundle modeled on $\pi_M : \mathsf{T}^*M \to M$ with the affine structure

$$\mathbf{d}\sigma_2(m) - \mathbf{d}\sigma_1(m) = \mathrm{d}(\sigma_2 - \sigma_1)(m).$$

This affine bundle is called the *phase bundle* of ζ. A section of $\mathsf{P}\zeta$ will be called an *affine one-form*.

Let $\alpha : M \to \mathsf{P}\mathbf{Z}$ be an affine one-form and let σ be a section of ζ. The differential $\mathrm{d}_m(\alpha - \mathbf{d}\sigma)$ does not depend on the choice of σ and will be called the *differential of* α *at* m. We will denote it by $\mathbf{d}\alpha(m)$ or by $\mathbf{d}_m\alpha$. The differential of an affine one-form $\alpha \in \mathsf{Sec}(\mathsf{P}\mathbf{Z})$ is an ordinary 2-form $\mathbf{d}\alpha \in \Omega^2(M)$. The corresponding *affine de Rham complex* looks now like

$$\mathsf{Sec}(\mathbf{Z}) \xrightarrow{\mathbf{d}} \mathsf{Sec}(\mathsf{P}\mathbf{Z}) \xrightarrow{\mathbf{d}} \Omega^2(M) \xrightarrow{\mathrm{d}} \Omega^3(M) \xrightarrow{\mathrm{d}} \cdots \qquad (22)$$

and consists of affine maps. This is an *affine complex* in this sense that its linear part is a complex of linear maps, so that we can define the corresponding cohomology. The linear part of (22) is a part of the standard de Rham complex (without its beginning consisting of the inclusion of $\mathbb{R}$ into $C^\infty(M)$). However, note that the cohomology of (22) can be defined without refereing to its linear part. Indeed, the problem is only with the first and the second cohomology space, since the rest is the standard de Rham complex. Denote the kernel (the inverse image of $\{0\}$) and the image of the affine map $\mathbf{d} : \mathsf{Sec}(\mathsf{P}\mathbf{Z}) \to \Omega^2(M)$ by Z_1 and B_2, respectively. Z_1 is an affine subspace of $\mathsf{Sec}(\mathsf{P}\mathbf{Z})$ and B_2 is a vector subspace of the kernel Z_2 of $\mathrm{d} : \Omega^2(M) \to \Omega^3(M)$. Moreover, the image B_1 of $\mathbf{d} : \mathsf{Sec}(\mathbf{Z}) \to \mathsf{Sec}(\mathsf{P}\mathbf{Z})$ is an affine subspace in Z_1. But the quotients of affine spaces are vector spaces, so that $H^1 = Z_1/B_1$ and $H^2 = Z_2/B_2$ are vector spaces. It is easy to see that we got nothing but the lacking first and second de Rham cohomology.

Recall that the bundle $\mathbf{Z}$ can be considered as a principal bundle with the structure group $(\mathbb{R}, +)$ and the fundamental vector field of this action we have denoted by $X_\mathbf{Z}$. Let us observe now that $\mathsf{P}\mathbf{Z}$ represents the principal connection bundle of $\mathbf{Z}$, i.e. there is a canonical identification of the affine space $\mathsf{Sec}(\mathsf{P}\mathbf{Z})$ of sections of $\mathsf{P}\mathbf{Z}$ with the affine space $\mathsf{PConn}(\mathbf{Z})$ of principal connections in $\mathbf{Z}$. We will identify the space of principal connections with the space of connection one-forms. In other words, $\mathsf{PConn}(\mathbf{Z})$ consists of those one-forms ν on $\mathbf{Z}$ which are $X_\mathbf{Z}$-invariant and $\nu(X_\mathbf{Z}) = 1$. Since we can add to ν the pull-backs of one-forms on M, both affine spaces are modeled on the space $\Omega^1(M)$ of one-forms on M.

Theorem 14. *There is a canonical isomorphism of affine spaces* $\mathsf{F} : \mathsf{Sec}(\mathsf{P}\mathbf{Z}) \to \mathsf{PConn}(\mathbf{Z})$, *with linear part being the identity on* $\Omega^1(M)$, *such that for any section* σ *of* $\mathbf{Z}$

$$\mathsf{F}_{\mathbf{d}\sigma} = \mathrm{d}\mathbf{F}_\sigma,$$

where $\mathbf{F}_\sigma(\sigma') = \sigma - \sigma'$. *Moreover,*

$$\mathrm{d}\mathsf{F}_\alpha = \zeta^*(\mathbf{d}\alpha),$$

so that the 2*-form* $\mathbf{d}\alpha \in \Omega^2(M)$ *is the curvature form of the connection* $\alpha \in \mathsf{Sec}(\mathsf{P}\mathbf{Z})$.

Proof. Indeed, since $X_{\mathbf{Z}}(\mathbf{F}_\sigma) = 1$, $\mathrm{d}\mathbf{F}_\sigma \in \mathsf{PConn}(\mathbf{Z})$. Moreover, for $f \in C^\infty(M)$,

$$\mathsf{F}_{\mathbf{d}(\sigma+f)} = \mathrm{d}(\mathbf{F}_\sigma + f \circ \zeta),$$

so that

$$\mathsf{F}_{\mathbf{d}\sigma+\mathrm{d}f} = \mathsf{F}_{\mathbf{d}\sigma} + \zeta^*\mathrm{d}f$$

and we can define F_α for arbitrary $\alpha \in \mathsf{Sec}(\mathsf{P}\mathbf{Z})$ by $\mathsf{F}_\alpha = \mathsf{F}_{\mathbf{d}\sigma} + \zeta^*(\alpha - \mathbf{d}\sigma)$. Finally,

$$\mathrm{d}\mathsf{F}_\alpha = \mathrm{d}(\mathsf{F}_{\mathbf{d}\sigma} + \zeta^*(\alpha - \mathbf{d}\sigma)) = \zeta^*\mathrm{d}(\alpha - \mathbf{d}\sigma) = \zeta^*\mathbf{d}\alpha.$$

Conversely, if $\nu \in \mathsf{PConn}(\mathbf{Z})$, then $\nu - \mathbf{d}\sigma$ is a vertical and $X_{\mathbf{Z}}$-invariant one-form on $\mathbf{Z}$ for any section σ of $\mathbf{Z}$, thus $\nu - \mathbf{d}\sigma = \zeta^*(\mu)$ for certain one-form μ on M. Then, $\nu = \mathsf{F}_\alpha$ for $\alpha = \mathbf{d}\sigma + \mu$. □

In local affine coordinates (x^a, s) on $\mathbf{Z}$ we have

$$\mathsf{F}_{\alpha_a(x)\mathrm{d}x^a} = \alpha_a(x)\mathrm{d}x^a - \mathrm{d}s.$$

As we have noticed, there is a distinguished affine space Z_1 of *closed* affine one-forms. It turns out that, like in the case of the cotangent bundle, they can be defined intrinsically as those sections of $\mathsf{P}\mathbf{Z}$ whose images are Lagrangian submanifolds with respect to a *canonical symplectic structure* on $\mathsf{P}\mathbf{Z}$ which is defined as follows.

For a chosen section σ of ζ we have isomorphisms

$$I_\sigma : \mathbf{Z} \to M \times \mathbb{R}, \qquad I_{\mathbf{d}\sigma} : \mathsf{P}\mathbf{Z} \to \mathsf{T}^*M \tag{23}$$

and for two sections σ, σ' the mappings $I_{\mathbf{d}\sigma}$ and $I_{\mathbf{d}\sigma'}$ differ by translation by $\mathrm{d}(\sigma - \sigma')$, i.e.

$$I_{\mathbf{d}\sigma'} \circ I_{\mathbf{d}\sigma}^{-1} : \mathsf{T}^*M \to \mathsf{T}^*M : \alpha_m \mapsto \alpha_m + \mathrm{d}(\sigma - \sigma')(m). \tag{24}$$

Now we use an affine property of the canonical symplectic form ω_M on the cotangent bundle: *translations in* T^*M *by a closed* 1*-form are symplectomorphisms*, to conclude that the two-form $I_{\mathbf{d}\sigma}^*\omega_M$, where ω_M is the canonical symplectic form on T^*M, does not depend on the choice of σ and therefore it is a canonical symplectic form on $\mathsf{P}\mathbf{Z}$. We will denote this form by $\omega_{\mathbf{Z}}$.

Theorem 15. *An affine* 1*-form* $\alpha \in \mathsf{Sec}(\mathsf{P}\mathbf{Z})$ *is closed if and only if* $\alpha(M)$ *is a Lagrangian submanifold of* $(\mathsf{P}\mathbf{Z}, \omega_{\mathbf{Z}})$.

Proof. Consider a section $\sigma \in \mathsf{Sec}(\mathbf{Z})$ and the corresponding isomorphism of affine bundles $I_{\mathbf{d}\sigma} : \mathsf{P}\mathbf{Z} \to \mathsf{T}^*M$. With respect to this isomorphism any affine one-form α corresponds to the true one-form $\alpha - \mathbf{d}\sigma$ on $M : I_{\mathbf{d}\sigma}(\alpha(m)) = \alpha(m) - \mathbf{d}\sigma(m)$. According to the well-known characterization, $\alpha - \mathbf{d}\sigma$ is closed if and only if $(\alpha - \mathbf{d}\sigma)(M)$ is a Lagrangian submanifod in $(\mathsf{T}^*M, \omega_M)$ so if and only if

$$\alpha(M) = I_{\mathbf{d}\sigma}^{-1}((\alpha - \mathbf{d}\sigma)(M))$$

is a Lagrangian submanifold of $(\mathsf{P}\mathbf{Z}, \omega_{\mathbf{Z}})$, since $I_{\mathbf{d}\sigma}$ is a symplectomorphism. But, by definition, $\mathrm{d}(\alpha - \mathbf{d}\sigma) = 0$ if and only if $\mathbf{d}\alpha = 0$. □

Remark. It is obvious that $\mathsf{P}\mathbf{Z}$ and $\mathsf{P}\bar{\mathbf{Z}}$ are equal as manifolds. Let σ be a section of $\mathbf{Z}$. The same mapping interpreted as a section of $\bar{\mathbf{Z}}$ will be denoted by $\bar{\sigma}$. Since $\sigma - \sigma' = \bar{\sigma}' - \bar{\sigma}$, the isomorphisms $I_{\mathbf{d}\sigma} : \mathsf{P}\mathbf{Z} \to \mathsf{T}^*M$ and $I_{\mathbf{d}\bar{\sigma}} : \mathsf{P}\bar{\mathbf{Z}} \to \mathsf{T}^*M$ are related by $I_{\mathbf{d}\sigma} = -I_{\mathbf{d}\bar{\sigma}}$. It follows that

$$\omega_{\bar{\mathbf{Z}}} = I_{\mathbf{d}\bar{\sigma}}^*\omega_M = -I_{\mathbf{d}\sigma}^*\omega_M = -\omega_{\mathbf{Z}}.$$

There is no canonical Liouville one-form on $\mathsf{P}\mathbf{Z}$ (in the standard sense) which is the potential of the canonical symplectic form $\omega_{\mathbf{Z}}$ but there is such a form in the affine sense. To define this Liouville one-form let us build another canonical affine bundle out of $\mathbf{Z}$.

We define another equivalence relation in the set of all pairs (m, σ). Two pairs (m, σ) and (m', σ') are equivalent if $m' = m$, $\sigma(m) = \sigma'(m)$, and $\mathrm{d}(\sigma' - \sigma)(m) = 0$. We can identify the equivalence class of (m, σ) with the first jet of the section σ with the source point m. We denote by $\mathsf{C}\mathbf{Z}$ the set of equivalence classes. The class of (m, σ) will be denoted by $\mathrm{c}\sigma(m)$ or by $\mathrm{c}_m\sigma$ and will be called the *contact element* of σ at m. We define a fiber bundle structure over M defining the projection $\mathsf{C}\zeta : \mathsf{C}\mathbf{Z} \to M$ by $\mathsf{C}\zeta(\mathrm{c}\sigma(m)) = m$. In other words, $\mathsf{C}\mathbf{Z}$ is the first-jet bundle $j^1(\zeta)$ of ζ. This fiber bundle is canonically an affine bundle modeled on $\gamma_M : \mathsf{T}^*M \oplus \mathbb{R} \to M$ with the affine structure defined by

$$\mathrm{c}\sigma_2(m) - \mathrm{c}\sigma_1(m) = (\mathrm{d}(\sigma_2 - \sigma_1)(m), \sigma_2(m) - \sigma_1(m)).$$

This affine bundle is called the *contact bundle* of ζ. The pair $(\mathsf{C}\mathbf{Z}, (0, 1_M))$ is a special affine bundle. It is easy to see $\mathsf{C}\mathbf{Z} = \mathsf{P}\mathbf{Z} \times^a{}_M \mathbf{Z}$ and that $\mathsf{C}\mathbf{Z}/\langle(0, 1_M)\rangle$ is canonically isomorphic to $\mathsf{P}\mathbf{Z}$ (we just identify the points m in the equivalence relation) so we have the associated AV-bundle with the canonical projection $\zeta_{\mathsf{C}\mathbf{Z}} : \mathsf{C}\mathbf{Z} \to \mathsf{P}\mathbf{Z}$.

There is also a canonical projection

$$\mu : \mathsf{C}\mathbf{Z} \to \mathbf{Z}, \qquad \mu(\mathrm{c}\sigma(m)) = \sigma(m), \tag{25}$$

which is a morphism of special affine bundles $\zeta_{\mathsf{C}\mathbf{Z}} : \mathsf{C}\mathbf{Z} \to \mathsf{P}\mathbf{Z}$ and $\zeta : \mathbf{Z} \to M$ over the projection $\zeta : \mathsf{P}\mathbf{Z} \to M$ on the level of base manifolds and there is a well-defined pull-back of sections of ζ to sections of $\zeta_{\mathsf{C}\mathbf{Z}}$. Now we can define a section $\theta_{\mathbf{Z}}$ of $\mathsf{P}\zeta_{\mathsf{C}\mathbf{Z}} : \mathsf{P}\mathsf{C}\mathbf{Z} \to \mathsf{P}\mathbf{Z}$ by

$$\theta_{\mathbf{Z}}(p) = \mathbf{d}_m\mu^*\sigma, \tag{26}$$

$\mathsf{P}\zeta(p) = m$, where σ is a section of ζ which represents $p \in \mathsf{P}_m\mathbf{Z}$. In other words, for any section $\sigma \in \mathsf{Sec}(\mathbf{Z})$

$$\theta_{\mathbf{Z}}(\mathbf{d}\sigma(m)) = \mathbf{d}(\mu^*\sigma)(\mathbf{d}\sigma(m)).$$

The affine one-form $\theta_{\mathbf{Z}}$ is called the *Liouville affine form* of $\mathsf{C}\mathbf{Z}$ and defines the *canonical contact structure* of $\mathsf{C}\mathbf{Z}$. This affine one-form over $\mathsf{P}\mathbf{Z}$ is a potential for the canonical symplectic form on $\mathsf{P}\mathbf{Z}$.

Theorem 16. $\mathbf{d}\theta_{\mathbf{Z}} = \omega_{\mathbf{Z}}$.

Proof. Let us take a section σ_0 of $\mathbf{Z}$. Using the identification $I_{\mathrm{c}\sigma_0} : \mathsf{C}\mathbf{Z} \to \mathsf{T}^*M \times \mathbb{R}$ we identify the AV-bundle $\zeta_{\mathsf{C}\mathbf{Z}} : \mathsf{C}\mathbf{Z} \to \mathsf{P}\mathbf{Z}$ with the trivial bundle $pr_{\mathsf{T}^*M} : \mathsf{T}^*M \times \mathbb{R} \to \mathsf{T}^*M$. With this identification sections of $\zeta_{\mathsf{C}\mathbf{Z}}$ are functions on T^*M and $\mu^*\sigma_0 = 0$ is a distinguished section of $\zeta_{\mathsf{C}\mathbf{Z}}$, so that sections of $\mathsf{PC}\mathbf{Z}$ are standard one-forms on T^*M with the standard de Rham differential. Moreover, sections of $\mathbf{Z}$ are represented by functions on M, μ^* is represented by π_M^*, and the symplectic form $\omega_{\mathbf{Z}}$ is represented by the standard symplectic form ω_M on T^*M.

Take a section σ of $\mathbf{Z}$ understood as a function on M. By definition, $\theta_{\mathbf{Z}}(\mathrm{d}\sigma(m)) = \mathrm{d}(\pi_M^*\sigma)(\mathrm{d}\sigma(m))$ which means that $\theta_{\mathbf{Z}}$ is represented by the true Liouville one-form θ_M on T^*M. Hence, $\mathbf{d}\theta_{\mathbf{Z}} = \omega_{\mathbf{Z}}$. □

Remark. Note that the above proof does not imply that we can define a true canonical Liouville one-form on $\mathsf{P}\mathbf{Z}$. Indeed, it is easy to see that the change of the initial section σ_0 into σ_0' with $\sigma_0' = \sigma_0 + f$ results, for the trivialization given by σ_0, in translation of the LiouvilleLiouville one-form: $\theta_M \mapsto \theta_M - \pi_M^*\mathrm{d}f$. Thus, the true Liouville one-form on T^*M has no affine meaning (but its exterior derivative has such a meaning), since it is not invariant by translations by $\mathrm{d}\pi_M^*(f)$). We put a geometrical meaning to the transformation rules of the Liouville one-form defining its affine version. This explains perhaps better what an affine one-form is.

The affine Liouville one-form $\theta_{\mathbf{Z}}$ can be interpreted as a canonical principal connection on the principal bundle $\zeta_{\mathsf{C}\mathbf{Z}} : \mathsf{C}\mathbf{Z} \to \mathsf{P}\mathbf{Z}$, thus as a canonical one-form $\eta_{\mathsf{C}\mathbf{Z}} = \mathsf{F}_{\theta_{\mathbf{Z}}}$ on $\mathsf{C}\mathbf{Z}$. In any trivialization $I_{\mathrm{c}\sigma} : \mathsf{C}\mathbf{Z} \to \mathsf{T}^*M \times \mathbb{R}$ and the standard Darboux coordinates (x^a, p_b, s) on $\mathsf{T}^*M \times \mathbb{R}$ the affine Liouville one-form has the standard expression $\theta_{\mathbf{Z}} = p_a\,\mathrm{d}x^a$, so $\eta_{\mathsf{C}\mathbf{Z}}$ looks like the canonical contact form: $\eta_{\mathsf{C}\mathbf{Z}} = p_a\,\mathrm{d}x^a - \mathrm{d}s$. It can be also seen directly that the canonical contact form $\eta_M = p_a\,\mathrm{d}x^a - \mathrm{d}s$ on $\mathsf{T}^*M \times \mathbb{R}$ is affine in this sense that it is invariant with respect to translations of $\mathsf{T}^*M \times \mathbb{R}$ by first jets of functions . We will call $\eta_{\mathsf{C}\mathbf{Z}}$ the *canonical contact form* on $\mathsf{C}\mathbf{Z}$. Like every contact form, it induces a (contact) Jacobi bracket $\{\cdot,\cdot\}_{\mathsf{C}\mathbf{Z}}$ on $\mathsf{C}\mathbf{Z}$ which in the above local coordinates reads

$$\{f, g\}_{\mathsf{C}\mathbf{Z}} = \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial x^a} - \frac{\partial g}{\partial p_a}\frac{\partial f}{\partial x^a} + \left(p_a\frac{\partial f}{\partial p_a} - f\right)\frac{\partial g}{\partial s} - \left(p_a\frac{\partial g}{\partial p_a} - g\right)\frac{\partial f}{\partial s}. \tag{27}$$

The corresponding Jacobi structure on $\mathsf{C}\mathbf{Z}$ in this trivialization takes the form $(\Lambda_M + \Delta_{\mathsf{T}^*M}\wedge \partial_s, -\partial_s)$, where $\Lambda_M = \partial_{p_a}\wedge\partial_{x^a}$ is the canonical Poisson tensor on T^*M associated with the syplectic form ω_M and $\Delta_{\mathsf{T}^*M} = p_a\partial_{p_a}$ is the Liouville vector field on T^*M (both regarded as tensor fields on $\mathsf{T}^*M \times \mathbb{R}$). Again, this Jacobi structure has an affine flavor, since the tensors are invariant with respect to translations in $\mathsf{T}^*M \times \mathbb{R}$ by first jets of functions.

9. Lie algebroids associated with AV-bundles

The principal bundle structure of $\mathbf{Z}$ represented by the vector field $X_{\mathbf{Z}}$ induces additional structures on functions, vector fields and, in general, differential operators on Z. For any manifold N denote by $\mathcal{X}(N)$ (resp., $\mathcal{D}^1(N)$) the space of all vector fields (resp., the space of all linear first-order differential operators) on N, i.e. acting on $C^\infty(N)$. Clearly, $\mathcal{X}(N) = \mathsf{Sec}(\mathsf{T}N)$ and $\mathcal{D}^1(N)$ is the space of sections of the bundle $\mathsf{L}N = \mathsf{T}N \oplus \mathbb{R}$ of linear first-order differential operators on N with the obvious action of $(X+h) \in \mathsf{Sec}(\mathsf{L}N) = \mathcal{X}(N) \oplus C^\infty(N)$ on functions on N given by $(X+h)(f) = X(f) + hf$.

Let us fix an AV-bundle $\mathbf{Z}$ over M. The space $\mathrm{Pol}^n(\mathbf{Z})$ of polynomials of order $\leq n$ on $\mathbf{Z}$ is defined as the space of those smooth functions f on $\mathbf{Z}$ for which $X_{\mathbf{Z}}^{n+1}(f) = 0$, so that we have the filtration $\mathrm{Pol}(\mathbf{Z}) = \cup_n \mathrm{Pol}^n(\mathbf{Z})$ of the algebra of all polynomials. Note that in the affine case we have only the filtration and no canonical graduation of $\mathrm{Pol}(\mathbf{Z})$. In particular, the space $\mathrm{Pol}^0(\mathbf{Z})$ is just the algebra $\mathrm{Bas}(\mathbf{Z})$ of basic functions on $\mathbf{Z}$, i.e. functions that are constant along fibers (it will be often identified with the algebra of smooth functions on M) and $\mathrm{Pol}^1(\mathbf{Z})$ is the space $\mathsf{Aff}(\mathbf{Z})$ of affine (along fibers) functions on $\mathbf{Z}$.

We have also natural subalgebras of the Lie algebra $\mathcal{X}(\mathbf{Z})$ of all vector fields on $\mathbf{Z}$. The Lie algebra $\tilde{\mathcal{X}}(\mathbf{Z})$ of invariant vector fields on $\mathbf{Z}$ consists of those vector fields X for which $[X_{\mathbf{Z}}, X] = 0$. It is easy to see that, in local affine coordinates (x^a, s) on $\mathbf{Z}$, invariant vector fields have the form

$$X = f_a(x)\partial_{x^a} - g(x)\partial_s,$$

where the functions f_a, g are basic. Vector fields from $\tilde{\mathcal{X}}(\mathbf{Z})$ can be viewed as sections of the vector bundle $\tilde{\mathsf{T}}\mathbf{Z} = \mathsf{T}\mathbf{Z}/\mathbb{R}$ which is the vector bundle over M of orbits of the tangent lift ϕ_* of the $(\mathbb{R}, +)$ action ϕ on $\mathbf{Z}$. Since the vector field $X_{\mathbf{Z}}$ is invariant, it can be understood as a distinguished section of $\tilde{\mathsf{T}}\mathbf{Z}$, so $\tilde{\mathsf{T}}\mathbf{Z}$ is canonically a special vector bundle. This is just the Atiyah vector bundle (and canonically a Lie algebroid) associated with the $\mathbb{R}$-principal bundle $\mathbf{Z}$.

There is another natural subalgebra of the Lie algebra $\mathcal{X}(\mathbf{Z})$ of all smooth vector fields on $\mathbf{Z}$, namely the subalgebra $\mathcal{X}_{ah}(\mathbf{Z})$ of *affine-homogeneous vector fields*, i.e. those vector fields X which preserve the filtration: $X(\mathrm{Pol}^n(\mathbf{Z})) \subset \mathrm{Pol}^n(\mathbf{Z})$. Of course, $\tilde{\mathcal{X}}(\mathbf{Z}) \subset \mathcal{X}_{ah}(\mathbf{Z})$, since invariant vector fields lower the filtration: $X(\mathrm{Pol}^n(\mathbf{Z})) \subset \mathrm{Pol}^{n-1}(\mathbf{Z})$. Again, the space $\mathcal{X}_{ah}(\mathbf{Z})$ is the space of sections of certain vector bundle $\check{\mathsf{L}}\mathbf{Z}$ over M which can be identified with the bundle $\tilde{\mathsf{T}}\mathbf{Z} \oplus^{sv}{}_M \mathbf{Z}^\dagger = (\tilde{\mathsf{T}}\mathbf{Z} \oplus_M \mathbf{Z}^\dagger)/\langle X_{\mathbf{Z}} - 1_{\mathbf{Z}}\rangle$, i.e. the special direct sum of the special vector bundles $\tilde{\mathsf{T}}\mathbf{Z}$ and $\mathbf{Z}^\dagger$. Indeed, it is easy to see that the class $Y \oplus^{sv} \varphi \in \mathsf{Sec}(\tilde{\mathsf{T}}\mathbf{Z} \oplus^{sv}{}_M \mathbf{Z}^\dagger)$, where $Y = f_a(x)\partial_{x^a} - g(x)\partial_s$ and $\varphi(x,s) = \alpha(x)s + \beta_0(x)$, is represented by $f_a(x)\partial_{x^a} - (\alpha(x)s + \beta(x))\partial_s$ with $\beta(x) = \beta_0(x) + g(x)$, so the vector field $\mathring{\mathbf{D}}_{X \oplus^{sv} \varphi} = X + \varphi X_{\mathbf{Z}}$ gives such an identification.

Similarly, there is a natural subalgebra of the Lie algebra $\mathcal{D}^1(\mathbf{Z}) = \mathcal{D}^1(Z)$, the subalgebra $\mathcal{D}^1_{ah}(\mathbf{Z})$ of *affine-homogeneous first-order differential operators*, consisting of those $D \in \mathcal{D}^1(\mathbf{Z})$ which preserve the filtration: $D(\mathrm{Pol}^n(\mathbf{Z})) \subset \mathrm{Pol}^n(\mathbf{Z})$. Note that $\mathcal{D}^1_{ah}(\mathbf{Z})$ is canonically a $Bas(\mathbf{Z}) \simeq C^\infty(M)$-module.

It is easy to see the following proposition.

Proposition 2. *There is a canonical splitting* $\mathcal{D}^1_{ah}(\mathbf{Z}) = \mathcal{X}_{ah}(\mathbf{Z}) \oplus \mathrm{Bas}(\mathbf{Z})$. *Moreover, a vector field* X *on* $\mathbf{Z}$ *is affine-homogeneous if and only if* $[X_{\mathbf{Z}}, [X_{\mathbf{Z}}, X]] = 0$ *and* $[X_{\mathbf{Z}}, X]$ *is vertical. In local affine coordinates* (x^a, s) *on* $\mathbf{Z}$, *affine-homogeneous vector fields have precisely the form*

$$X = f_a(x)\partial_{x^a} - (\alpha(x)s + \beta(x))\partial_s,$$

and affine-homogeneous first-order differential operators the form

$$D = f_a(x)\partial_{x^a} - (\alpha(x)s + \beta(x))\partial_s + \gamma(x).$$

Note that the vector fields from $\mathcal{X}_{ah}(\mathbf{Z})$ are projectable and the vector field $X = f_a(x)\partial_{x^a} - (\alpha(x)s + \beta(x))\partial_s$ projects onto the vector field $\mathring{X} = f_a(x)\partial_{x^a}$ on M. Before finding an appropriate bundle whose sections form $\mathcal{D}^1_{ah}(\mathbf{Z})$ let us observe that the canonical Jacobi bracket $\{\cdot, \cdot\}_{\mathbf{Z}}$ applied to affine functions $\varphi, \psi \in \mathsf{Aff}(\mathbf{Z})$ gives a basic function. Indeed, since $X^2_{\mathbf{Z}}(\varphi) = X^2_{\mathbf{Z}}(\psi) = 0$, we have

$$X_{\mathbf{Z}}(\{\varphi, \psi\}_{\mathbf{Z}}) = X_{\mathbf{Z}}(\varphi X_{\mathbf{Z}}(\psi) - \psi X_{\mathbf{Z}}(\varphi)) = 0.$$

Recall that the map $\mathbf{F}$ identifies sections of $\hat{\mathbf{Z}}$ with $\mathsf{Aff}(\mathbf{Z})$. In particular, we can identify sections σ of $\mathbf{Z}$ with affine functions $\mathbf{F}_\sigma$ which satisfy $X_{\mathbf{Z}}(\mathbf{F}_\sigma) = 1$, so that $\mathsf{Sec}(\mathbf{Z}) \simeq \{\varphi \in \mathsf{Aff}(\mathbf{Z}) : X_{\mathbf{Z}}(\varphi) = 1_{\mathbf{Z}}\}$. We have the following bundle version of Theorem 9 obtained just fiberwise.

Theorem 17. *For all* $\phi \in \mathsf{Aff}(\mathbf{Z}) = \mathsf{Sec}(\mathbf{Z}^\dagger)$ *and all* $u \in \mathsf{Sec}(\hat{\mathbf{Z}}) = \mathsf{Sec}((\mathbf{Z}^\dagger)^*)$:

$$\{\phi, \mathbf{F}_u\}_{\mathbf{Z}} = \langle \phi, u \rangle. \tag{28}$$

There is a 'Hamiltonian map'

$$\mathsf{Aff}(\mathbf{Z}) \ni \varphi \mapsto \mathbf{D}_\varphi = \varphi X_{\mathbf{Z}} - X_{\mathbf{Z}}(\varphi) = \{\varphi, \cdot\}_{\mathbf{Z}} \in \mathcal{D}^1(\mathbf{Z})$$

with the property

$$\mathbf{D}_\varphi(\mathbf{F}_u) = \{\varphi, \mathbf{F}_u\}_{\mathbf{Z}} = \langle \varphi, u \rangle$$

for $\varphi \in \mathsf{Aff}(\mathbf{Z})$, $u \in \mathsf{Sec}(\hat{\mathbf{Z}})$. Therefore we can consider $\mathbf{Z}^\dagger$ as embedded in $\mathcal{D}^1_{ah}(\mathbf{Z})$. For a section σ of $\hat{\mathbf{Z}}$ we will write shortly $\mathbf{D}_\sigma$ instead of $\mathbf{D}_{\mathbf{F}_\sigma}$.

In local affine coordinates, the Jacobi bracket of affine functions on $\mathbf{Z}$ takes the form

$$\{\alpha(x)s + \beta(x), \alpha'(x)s + \beta'(x)\}_{\mathbf{Z}} = \alpha(x)\beta'(x) - \alpha'(x)\beta(x)$$

and the differential operator associated to $\alpha(x)s + \beta(x) \in \mathsf{Aff}(\mathbf{Z})$ reads

$$\mathbf{D}_{\alpha(x)s+\beta(x)} = \alpha(x) - (\alpha(x)s + \beta(x))\partial_s. \tag{29}$$

Now, we can extend the map $\mathbf{D}$ to sections of the bundle $\mathsf{R}\mathbf{Z} = \check{\mathsf{L}}\mathbf{Z} \oplus^{sv}{}_M \mathbf{Z}^\dagger = \tilde{\mathsf{T}}\mathbf{Z} \oplus^{sv}{}_M \mathbf{Z}^\dagger \oplus^{sv}{}_M \mathbf{Z}^\dagger$ by

$$\mathbf{D}_{X \oplus^{sv} \varphi \oplus^{sv} \psi} = X + \varphi X_{\mathbf{Z}} + \mathbf{D}_\psi.$$

It is easy to see that this gives the identification of sections of $\mathsf{R}\mathbf{Z}$ with $\mathcal{D}^1_{ah}(\mathbf{Z})$. In local affine coordinates,

$$\mathbf{D}_R = f_a(x)\partial_{x^a} - ((\alpha(x) + \alpha'(x))s + g(x) + \beta(x) + \beta'(x))\partial_s + \alpha(x),$$

where $R = (f_a(x)\partial_{x^a} - g(x)\partial_s)\oplus^{sv}(\alpha'(x)s + \beta'(x))\oplus^{sv}(\alpha(x)s + \beta(x))$. It is obvious that the commutator bracket of first-order differential operators induces a Lie algebroid structure on $\mathsf{R}\mathbf{Z}$ with the anchor $(X\oplus^{sv}\varphi\oplus^{sv}\psi)^0 = \mathring{X}$. In local affine coordinates:

$$\begin{aligned}
&[f_a(x)\partial_{x^a} - (\alpha(x)s + \beta(x))\partial_s + \gamma(x),\, f'_a(x)\partial_{x^a} - (\alpha'(x)s + \beta'(x))\partial_s + \gamma'(x)]\\
&\quad = \left(f_b(x)\frac{\partial f'_a}{\partial x^b}(x) - f'_b(x)\frac{\partial f_a}{\partial x^b}(x)\right)\partial_{x^a} - \left(\left(f_a(x)\frac{\partial \alpha'}{\partial x^a}(x) - f'_a(x)\frac{\partial \alpha}{\partial x^a}(x)\right)\right)s\\
&\qquad + f_a(x)\frac{\partial \beta'}{\partial x^a}(x) - f'_a(x)\frac{\partial \beta}{\partial x^a}(x) + \alpha(x)\beta'(x) - \alpha'(x)\beta(x))\partial_s\\
&\qquad + \left(f_a(x)\frac{\partial \gamma'}{\partial x^a}(x) - f'_a(x)\frac{\partial \gamma}{\partial x^a}(x)\right)
\end{aligned}$$

and

$$(f_a(x)\partial_{x^a} - (\alpha(x)s + \beta(x))\partial_s + \gamma(x))^0 = f_a(x)\partial_{x^a}.$$

Writing $X = f_a(x)\partial_{x^a}$ and representing $f_a(x)\partial_{x^a} - (\alpha(x)s + \beta(x))\partial_s + \gamma(x)$ by $(X, \alpha, \beta, \gamma)$ we can write shortly

$$\begin{aligned}
&[(X, \alpha, \beta, \gamma), (X', \alpha', \beta', \gamma')]\\
&\quad = ([X, X'], X(\alpha') - X'(\alpha), X(\beta') - X'(\beta) + \alpha\beta' - \alpha'\beta, X(\gamma') - X'(\gamma)).
\end{aligned} \tag{30}$$

Note that the distinguished sections $X_{\mathsf{R}\mathbf{Z}} = -\partial_s$ and $I_{\mathsf{R}\mathbf{Z}} = 1$ are in this Lie algebroid *ideal sections*, i.e. these sections are nowhere-vanishing and the sections of the one-dimensional subbundles generated by $X_{\mathsf{R}\mathbf{Z}}$ and $I_{\mathsf{R}\mathbf{Z}}$ are Lie ideals with respect to the Lie algebroid bracket. A special vector bundle (E, X_0) equipped with a Lie algebroid structure such that X_0 is an ideal section we call an *ideal-special Lie algebroid*. An ideal-special Lie algebroid for which X_0 is a central section, i.e. X_0 commutes with any section with respect to the Lie algebroid bracket, we call a *special Lie algebroid*. It is easy to see that ideal-sections define canonically 1-cocycles for the corresponding Lie algebroids.

Proposition 3. *If X_0 is an ideal section of a Lie algebroid on the vector bundle E of rank > 1 over M, then there is a closed* '1-*form*' $\phi_{X_0} \in \mathsf{Sec}(E^*)$ *such that* $[Y, X_0] = \langle Y, \phi_{X_0}\rangle X_0$.

Proof. Since $[X_0, fY] = f[X_0, Y] + \rho(X_0)(f)Y$ and X_0 generates a Lie ideal, the anchor $\rho(X_0)$ vanishes if only $\mathrm{rank}(E) > 1$. Thus $[Y, X_0] = \Phi(Y)X_0$ for certain function $\Phi(Y)$ which linearly depends on $Y \in \mathsf{Sec}(E)$ and $\Phi(fY) = f\Phi(Y)$, so $\Phi(Y) = \langle Y, \phi\rangle$ for certain $\phi \in \mathsf{Sec}(E^*)$. The 'one-form' ϕ is closed with respect to the Lie algebroid de Rham differential, since, due to the Jacobi identity, $\Phi([Y_1, Y_2]) = \rho(Y_1)(\Phi(Y_2)) - \rho(Y_2)(\Phi(Y_1))$. □

For the Lie algebroid $\mathsf{R}\mathbf{Z}$ denote $\phi_{X_{\mathsf{RZ}}}$ by ϕ^0. In local affine coordinates, $\langle R, \phi^0\rangle = \alpha$ for $R = f_a(x)\partial_{x^a} - (\alpha(x)s + \beta(x))\partial_s + \gamma(x)$, so ϕ^0 is nowhere-vanishing. There is another canonical nowhere-vanishing closed 'one-form' ϕ^1 on $\mathsf{R}\mathbf{Z}$ induced by the decomposition $\mathcal{D}^1_{ah}(\mathbf{Z}) = \mathcal{X}_{ah}(\mathbf{Z}) \oplus \mathrm{Bas}(\mathbf{Z})$, namely $\langle \phi^1, R\rangle = \gamma$. Note that the form $\phi_{I_{\mathsf{RZ}}}$ is identically zero.

The bundles $\tilde{\mathsf{T}}\mathbf{Z}$ and $\check{\mathsf{L}}\mathbf{Z}$ are subbundles of $\mathsf{R}\mathbf{Z}$ characterized by $\phi^0 = \phi^1 = 0$ and $\phi^1 = 0$, respectively. On the level of realizations we have $\tilde{\mathsf{T}}\mathbf{Z} = \tilde{\mathsf{T}}\mathbf{Z} \oplus^{sv}{}_M \mathbf{I} \subset \check{\mathsf{L}}\mathbf{Z}$ and $\check{\mathsf{L}}\mathbf{Z} = \check{\mathsf{L}}\mathbf{Z} \oplus^{sv}{}_M \mathbf{I} \subset \mathsf{R}\mathbf{Z}$. Of course, $\tilde{\mathsf{T}}\mathbf{Z}$ and $\check{\mathsf{L}}\mathbf{Z}$ are Lie subalgebroids of $\mathsf{R}\mathbf{Z}$ in every natural sense. Thus we have the chain $\tilde{\mathsf{T}}\mathbf{Z} \subset \check{\mathsf{L}}\mathbf{Z} \subset \mathsf{R}\mathbf{Z}$ of Lie algebroids over M, canonically associated with $\mathbf{Z}$, whose Lie algebras of sections are $\tilde{\mathcal{X}}(\mathbf{Z})$, $\mathcal{X}_{ah}(\mathbf{Z})$, and $\mathcal{D}^1_{ah}(\mathbf{Z})$, respectively. The bundle $\mathbf{Z}^\dagger$ is the kernel of the anchor map in $\check{\mathsf{L}}\mathbf{Z}$, so $\mathbf{Z}^\dagger$ is canonically a Lie algebroid with the trivial anchor. It is easy to see that the Lie algebroid bracket on $\mathsf{Sec}(\mathbf{Z}^\dagger) = \mathsf{Aff}(\mathbf{Z})$ is given by

$$[\varphi, \psi] = \{\varphi, \psi\}_{\mathbf{Z}}. \tag{31}$$

Remark. The embedding of $\mathcal{D}^1_{ah}(\mathbf{Z})$ into $\mathcal{D}^1(\mathbf{Z})$ corresponds also to a Lie algebroid morphism from $\mathsf{R}\mathbf{Z}$ into $\mathsf{L}\mathbf{Z}$. This morphism, however, is of a different kind than morphism which are considered usually and which are associated with the standard morphisms of vector bundles, and it is represented by a relation, not a map. This kind of morphisms is the Lie algebroid version of the Zakrzewski's morphisms of groupoids (see [28]). The Zakrzewski's morphisms of groupoids lead to satisfactory functors into C^*-algebras (cf. [19]).

We can embed $\tilde{\mathsf{T}}\mathbf{Z} \oplus^{sv}{}_M \mathbf{Z}^\dagger$ into $\tilde{\mathsf{T}}\mathbf{Z} \oplus^{sv}{}_M \mathbf{Z}^\dagger \oplus^{sv}{}_M \mathbf{Z}^\dagger \simeq \mathsf{R}\mathbf{Z}$ putting $\mathbf{I}$ not on the third place but on the second. The resulting subbundle of $\mathsf{R}\mathbf{Z}$ we will denote $\tilde{\mathsf{L}}\mathbf{Z}$. It can be described as the one determined by the equation $\phi^1 - \phi^0 = 0$ and therefore it is also a Lie subalgebroid of $\mathsf{R}\mathbf{Z}$ like every kernel of a closed nowhere-vanishing one-form. The induced Lie algebroid structure on $\tilde{\mathsf{T}}\mathbf{Z} \oplus^{sv}{}_M \mathbf{Z}^\dagger$ reads

$$[X \overset{sv}{\oplus} \varphi, X' \overset{sv}{\oplus} \varphi'] = [X, X'] \overset{sv}{\oplus} (X(\varphi') - X'(\varphi) + \{\varphi, \varphi'\}_{\mathbf{Z}})$$

and it is the same as the one obtained from the identification of $\tilde{\mathsf{T}}\mathbf{Z} \oplus^{sv}{}_M \mathbf{Z}^\dagger$ with $\check{\mathsf{L}}\mathbf{Z}$. In other words, $\check{\mathsf{L}}\mathbf{Z}$ and $\tilde{\mathsf{L}}\mathbf{Z}$ are isomorphic Lie algebroids differently placed in $\mathsf{R}\mathbf{Z}$. The sections of $\tilde{\mathsf{L}}\mathbf{Z}$ are first-order operators on $\mathbf{Z}$ having in local affine coordinates the form

$$D = f_a(x)\partial_{x^a} - (\alpha(x)s + \beta(x))\partial_s + \alpha(x).$$

The natural isomorphism with $\check{\mathsf{L}}\mathbf{Z}$ is just the restriction of the anchor map on $\mathsf{L}\mathbf{Z} = \mathsf{T}\mathbf{Z} \oplus \mathbb{R}$, i.e. $D \mapsto \mathring{D}$, where

$$\mathring{D} = f_a(x)\partial_{x^a} - (\alpha(x)s + \beta(x))\partial_s.$$

10. Affine derivations and affine first-order differential operators

Let us fix an AV-bundle $\zeta : \mathbf{Z} \to M$. In the standard differential geometry the phase and the contact bundles T^*M and $\mathsf{T}^*M \oplus \mathbb{R}$ are representing objects for derivations and linear first-order differential operators on $C^\infty(M)$, i.e. on sections of the trivial vector bundle $M \times \mathbb{R}$. By analogy, in AV-differential geometry by the *bundle of affine derivations* on $\mathbf{Z}$ (resp., the *bundle of affine first-order differential operators* on $\mathbf{Z}$) with values in an affine bundle A we understand the affine bundle $\mathsf{Aff}_M(\mathsf{P}\mathbf{Z}; A)$ (resp., $\mathsf{Aff}_M(\mathsf{C}\mathbf{Z}; A)$). Thus the affine space $\mathsf{ADer}(\mathbf{Z}; A)$ of *affine derivations* (resp., the space $\mathsf{ADO}^1(\mathbf{Z}; A)$ of *affine first-order differential operators*) on $\mathbf{Z}$ with values in A is the space of sections of this bundle. We have an obvious action $\sigma \mapsto D(\sigma)$ of $\mathsf{D} \in \mathsf{Aff}(\mathsf{P}\mathbf{Z}; A)$ (resp. $\mathsf{D} \in \mathsf{Aff}(\mathsf{C}\mathbf{Z}; A)$) on sections σ of $\mathbf{Z}$ by $D(\sigma) = \mathsf{D}(\mathbf{d}\sigma)$ (resp., $D(\sigma) = \mathsf{D}(\mathrm{c}\sigma)$). In the case $A = M \times \mathbb{R}$ we speak just about affine derivations (resp., affine first-order operators) on $\mathbf{Z}$ and denote the (linear) space $\mathsf{ADer}(\mathbf{Z}; \mathbb{R}) = \mathsf{Sec}(\mathsf{Aff}_M(\mathsf{P}\mathbf{Z}; \mathbb{R})) = \mathsf{Sec}(\mathsf{P}\mathbf{Z}^\dagger)$ (resp. $\mathsf{ADO}^1(\mathbf{Z}; \mathbb{R}) = \mathsf{Sec}(\mathsf{Aff}_M(\mathsf{C}\mathbf{Z}; \mathbb{R})) = \mathsf{Sec}(\mathsf{C}\mathbf{Z}^\dagger)$) simply by $\mathsf{ADer}(\mathbf{Z})$ (resp., $\mathsf{ADO}^1(\mathbf{Z})$). It is obvious by definition that the linear parts of affine derivations (resp. differential operators) are true derivations (resp. differential operators) on $C^\infty(M)$. It is also clear that these concepts can be extended naturally to a concept of a differential operator of arbitrary order. In this sense, the affine space $\mathsf{ADO}^0(\mathbf{Z}; A)$ of *affine differential operators of order 0* on $\mathbf{Z}$ with values in A is the space of sections of $\mathsf{Aff}_M(\mathbf{Z}, A)$, so the differential operators of order 0 with values in $\mathbb{R}$ are sections of $\mathbf{Z}^\dagger$.

To understand better the structure of the bundles of derivations and first-order differential operators let interpret them as certain bundles constructed out of $\mathbf{Z}$ in the way in which derivations of $C^\infty(M)$ are interpreted as vector fields, i.e. sections of $\mathsf{T}M$. Given an AV-bundle $\mathbf{Z}$ let us consider the cotangent bundle $\mathsf{T}^*\mathbf{Z}$. The $(\mathbb{R}, +)$-action ϕ on $\mathbf{Z}$ can be lifted to an $(\mathbb{R}, +)$-action ϕ^* on $\mathsf{T}^*\mathbf{Z}$, $(\phi^*)_r = (\phi_{-r})^*$. The fundamental vector field of this action we denote by $X_{\mathsf{T}^*\mathbf{Z}}$. The orbits $[\alpha_{z_m}]$ of this action form a vector bundle over M which we denote by $\tilde{\mathsf{T}}^*\mathbf{Z}$. The sections of $\tilde{\mathsf{T}}^*\mathbf{Z}$ are represented by one-forms on $\mathbf{Z}$, invariant with respect to $X_{\mathsf{T}^*\mathbf{Z}}$. Moreover, there is a canonical decomposition $\mathsf{T}^*\mathbf{Z} = \tilde{\mathsf{T}}^*\mathbf{Z} \times_M \mathbf{Z}$ given by

$$\alpha_{z_m} \mapsto ([\alpha_{z_m}], z_m) \tag{32}$$

which shows that $\mathsf{T}^*\mathbf{Z}$ is canonically an affine bundle over M with respect to the projection $\zeta\circ\pi_{\mathbf{Z}}$. This is a special affine bundle modeled on $\tilde{\mathsf{T}}^*\mathbf{Z}\times\mathbf{I}$. In local coordinates (x^a, s) on $\mathbf{Z}$ and the adapted coordinates (x^a, s, p_a, p) on $\mathsf{T}^*\mathbf{Z}$, the lifted action reads $(\phi^*)_r(x^a, s, p_a, p) = (x^a, s + r, p_a, p)$ and $X_{\mathsf{T}^*\mathbf{Z}} = -\partial_s$. Hence, (x^a, p_a, p) represent coordinates on $\tilde{\mathsf{T}}^*\mathbf{Z}$ and the section $p_a = p_a(x)$, $p = p(x)$, represents the invariant one-form $p_a(x)\mathrm{d}x^a + p(x)\mathrm{d}s$ on $\mathbf{Z}$. The affine phase bundle $\mathsf{P}\mathbf{Z}$ can be identified with the affine subbundle of $\tilde{\mathsf{T}}^*\mathbf{Z}$ in obvious way:

$$\mathsf{P}\mathbf{Z} = \{[\alpha_{z_m}] \in \tilde{\mathsf{T}}^*\mathbf{Z} : \langle \alpha_{z_m}, X_{\mathbf{Z}}(z_m)\rangle_{z_m} = 1\}.$$

Hence, $\widehat{\mathsf{P}\mathbf{Z}} \simeq \tilde{\mathsf{T}}^*\mathbf{Z}$. The contact bundle $\mathsf{C}\mathbf{Z}$ is an affine subbundle of $\mathsf{T}^*\mathbf{Z} = \tilde{\mathsf{T}}^*\mathbf{Z}\times^a{}_M\mathbf{Z}$ being the affine product $\mathsf{P}\mathbf{Z}\times^a{}_M\mathbf{Z}$.

We can do a similar procedure with the tangent bundle and obtain

$$\mathsf{T}\mathbf{Z} = \tilde{\mathsf{T}}\mathbf{Z} \times_M \mathbf{Z}.$$

The vector field $X_{\mathbf{Z}}$ is invariant, so it serves as a distinguished section of $\tilde{\mathsf{T}}\mathbf{Z}$. Thus $\tilde{\mathsf{T}}\mathbf{Z}$ is canonically a special vector bundle. Since $\tilde{\mathsf{T}}\mathbf{Z}$ is dual to $\tilde{\mathsf{T}}^*\mathbf{Z}$, it is obvious that $\tilde{\mathsf{T}}\mathbf{Z}^\ddagger = \mathsf{P}\mathbf{Z}$ (or, equivalently that $(\mathsf{P}\mathbf{Z})^\dagger = \tilde{\mathsf{T}}\mathbf{Z}$), since sections of $\mathsf{P}\mathbf{Z}$ are considered as invariant one-forms ν on $\mathbf{Z}$ such that $\nu(X_{\mathbf{Z}}) = 1$. Hence, $\mathsf{ADer}(\mathbf{Z}) = \mathsf{Sec}(\tilde{\mathsf{T}}\mathbf{Z}) = \tilde{\mathcal{X}}(\mathbf{Z})$ is the space of invariant vector fields X on $\mathbf{Z}$ and their action on sections σ of $\mathbf{Z}$ is given by $X(\sigma)\circ\zeta = X(\mathbf{F}_\sigma)$. In local affine coordinates (x^a, s) on $\mathbf{Z}$ for which $X_{\mathbf{Z}} = -\partial_s$, we can write $X = f_a(x)\partial_{x^a} + g(x)X_{\mathbf{Z}}$, so that

$$X(\sigma) \circ \zeta = (f_a(x)\partial_{x^a} - g(x)\partial_s)(\sigma(x) - s) = f_a(x)\frac{\partial\sigma}{\partial x^a}(x) + g(x).$$

We will use the natural convention and denote the pull-back $f \circ \zeta$ of a function $f \in C^\infty(M)$ to a basic function on $\mathbf{Z}$ by $\mathbf{F}_f$. With this convention we can simply write $\mathbf{F}_{X(\sigma)} = X(\mathbf{F}_\sigma)$.

According to Theorem 4 (3), $\mathsf{C}\mathbf{Z}^\dagger$ equals

$$(\mathsf{P}\mathbf{Z})^\dagger \overset{sv}{\oplus}_M \mathbf{Z}^\dagger = \tilde{\mathsf{T}}\mathbf{Z} \overset{sv}{\oplus}_M \mathbf{Z}^\dagger,$$

so $\mathsf{ADO}^1(\mathbf{Z}) = \mathsf{Sec}(\tilde{\mathsf{T}}\mathbf{Z}\oplus^{sv}{}_M\mathbf{Z}^\dagger)$. The section $D = X\oplus^{sv}\varphi \in \mathsf{Sec}(\tilde{\mathsf{T}}\mathbf{Z}\oplus^{sv}{}_M\mathbf{Z}^\dagger)$ acts on $\sigma \in \mathsf{Sec}(\mathbf{Z})$ by $X(\sigma) = X(\sigma) + \varphi \circ \sigma$. We will identify this bundle with $\check{\mathsf{L}}\mathbf{Z}$, since we can interpret this action by $D(\sigma) \circ \zeta = (X + \mathbf{D}_\varphi)(\mathbf{F}_\sigma)$. In local affine coordinates D has the form

$$D = f_a(x)\partial_{x^a} - (\alpha(x)s + \beta(x))\partial_s + \alpha(x)$$

and its action on sections of $\mathbf{Z}$ reads

$$\mathbf{F}_{D(\sigma)} = D(\sigma) \circ \zeta = D(\sigma(x) - s) = f_a(x)\frac{\partial\sigma}{\partial x^a}(x) + \alpha(x)\sigma(x) + \beta(x).$$

In view of Corollary 1 (Section 7),

$$\mathsf{ADer}(\mathbf{Z}; \mathbf{Z}) = \mathsf{Sec}(\tilde{\mathsf{T}}\mathbf{Z} \boxtimes_M \mathbf{Z}) = (\tilde{\mathsf{T}}\mathbf{Z} \overset{a}{\times}_M \mathbf{Z})/\langle X_{\mathbf{Z}} - 1_M\rangle.$$

Section $X \boxtimes \sigma' \in \mathsf{Sec}(\tilde{\mathsf{T}}\mathbf{Z} \boxtimes_M \mathbf{Z})$ acts on $\sigma \in \mathsf{Sec}(\mathbf{Z})$ by $(X \boxtimes \sigma')(\sigma) = X(\sigma) + \sigma'$. The embedding $\mathbf{F}$ of $\mathbf{Z}$ into $\mathbf{Z}^\dagger$ induces the obvious embedding of $\tilde{\mathsf{T}}\mathbf{Z} \boxtimes_M \mathbf{Z}$ as an affine hyperbundle $\bar{\mathsf{T}}\mathbf{Z}$ in $\tilde{\mathsf{T}}\mathbf{Z}\oplus^{sv}{}_M\mathbf{Z}^\dagger$. If we identify the last bundle with $\check{\mathsf{L}}\mathbf{Z}$, then $\bar{\mathsf{T}}\mathbf{Z}$ can be interpreted as an affine hyperbundle in $\check{\mathsf{L}}\mathbf{Z}$ and its sections can be interpreted as first-order differential operators on $\mathbf{Z}$ of the local form

$$\bar{X} = f_a(x)\partial_{x^a} + (s - \beta(x))\partial_s.$$

Their action on sections of $\mathbf{Z}$ is given by

$$\mathbf{F}_{\bar{X}(\sigma)}(x, s) = \bar{X}(\sigma(x) - s) = f_a(x)\frac{\partial\sigma}{\partial x^a}(x) + \beta(x) - s,$$

so

$$\bar{X}(\sigma)(x) = f_a(x)\frac{\partial\sigma}{\partial x^a}(x) + \beta(x).$$

Similarly as above, we get

$$\mathsf{Aff}(\mathsf{C}\mathbf{Z};\mathbf{Z}) = (\mathsf{C}\mathbf{Z})^\dagger \boxtimes_M \mathbf{Z} = \tilde{\mathsf{L}}\mathbf{Z} \boxtimes_M \mathbf{Z},$$

so that

$$\mathsf{ADO}^1(\mathbf{Z};\mathbf{Z}) = \mathsf{Sec}(\tilde{\mathsf{L}}\mathbf{Z} \boxtimes_M \mathbf{Z}).$$

An element $\bar{D} = D \boxtimes \sigma' \in \mathsf{Sec}(\tilde{\mathsf{L}}\mathbf{Z} \boxtimes_M \mathbf{Z})$ acts on $\sigma \in \mathsf{Sec}(\mathbf{Z})$ by $\bar{D}(\sigma) = D(\sigma) + \sigma'$. Again, the embedding $\mathbf{F} : \mathbf{Z} \to \mathbf{Z}^\dagger$ induces the obvious embedding of $\tilde{\mathsf{L}}\mathbf{Z} \boxtimes_M \mathbf{Z}$ as an affine hyperbundle of the vector bundle $\tilde{\mathsf{L}}\mathbf{Z} \oplus^{sv}{}_M \mathbf{Z}^\dagger = \mathsf{R}\mathbf{Z}$. This bundle we will denote shortly $\bar{\mathsf{L}}\mathbf{Z}$, so that, with respect to this identification, $\mathsf{ADO}^1(\mathbf{Z};\mathbf{Z})$ is the space of sections of $\bar{\mathsf{L}}\mathbf{Z}$, i.e. the space of first-order differential operators on $\mathbf{Z}$ of the local form

$$\bar{D} = f_a(x)\partial_{x^a} - ((\alpha(x) - 1)s + \beta(x))\partial_s + \alpha(x).$$

Then,

$$\mathbf{F}_{\bar{D}(\sigma)}(x, s) = f_a(x)\frac{\partial\sigma}{\partial x^a}(x) + \alpha(x)\sigma(x) + \beta(x) - s,$$

so

$$\bar{D}(\sigma)(x) = f_a(x)\frac{\partial\sigma}{\partial x^a}(x) + \alpha(x)\sigma(x) + \beta(x).$$

We can summarize all these observations as follows.

Theorem 18. *Let $\mathbf{Z}$ be an AV-bundle over M. There are subbundles of $\mathsf{R}\mathbf{Z}$: vector subbundles $\tilde{\mathsf{T}}\mathbf{Z}$, $\tilde{\mathsf{L}}\mathbf{Z}$, $\check{\mathsf{L}}\mathbf{Z}$ and affine subbundles: $\bar{\mathsf{T}}\mathbf{Z}$ modeled on $\tilde{\mathsf{T}}\mathbf{Z}$ and $\bar{\mathsf{L}}\mathbf{Z}$ modeled on $\tilde{\mathsf{L}}\mathbf{Z}$, characterized by $\phi^0 = \phi^1 = 0$, $\phi^1 - \phi^0 = 0$, $\phi^1 = 0$, $\phi^0 = 1$ and $\phi^1 = 0$, $\phi^1 - \phi^0 = 1$, respectively, such that*

(a) $\mathsf{ADer}(\mathbf{Z}) = \mathsf{Sec}(\tilde{\mathsf{T}}\mathbf{Z}) = \tilde{\mathcal{X}}(\mathbf{Z})$,
(b) $\mathsf{ADer}(\mathbf{Z};\mathbf{Z}) = \mathsf{Sec}(\bar{\mathsf{T}}\mathbf{Z})$,
(c) $\mathsf{ADO}^1(\mathbf{Z}) = \mathsf{Sec}(\tilde{\mathsf{L}}\mathbf{Z})$,
(d) $\mathsf{ADO}^1(\mathbf{Z};\mathbf{Z}) = \mathsf{Sec}(\bar{\mathsf{L}}\mathbf{Z})$,

and the action $\sigma \mapsto D(\sigma)$ of sections D of these bundles, regarded as elements of $\mathsf{Sec}(\mathsf{R}\mathbf{Z}) = \mathcal{D}^1_{ah}(\mathbf{Z})$, on sections σ of $\mathbf{Z}$ is given by

$$\mathbf{F}_{D(\sigma)} = D(\mathbf{F}_\sigma).$$

Remark. Of course, the vector bundle $\mathbf{Z}^\dagger$ (whose sections represent $\mathsf{ADO}^0(\mathbf{Z})$) and the affine bundle $\mathbf{Z}^\dagger \boxtimes_M \mathbf{Z}$ (whose sections represent $\mathsf{ADO}^0(\mathbf{Z};\mathbf{Z})$) are also subundles of $\mathsf{R}\mathbf{Z}$ contained in the kernel of the anchor map.

11. Canonical Lie affgebroids associated with AV-bundles

In the standard differential geometry the canonical Lie algebroid associated with a manifold M, or better, with the trivial bundle $M \times \mathbb{R}$, is $\mathsf{T}M$. With an AV-bundle $\mathbf{Z}$ we have associated the bundle $\tilde{\mathsf{T}}\mathbf{Z}$. Sections of $\tilde{\mathsf{T}}\mathbf{Z}$ are interpreted as affine derivations on sections of $\mathbf{Z}$. The bundle $\tilde{\mathsf{T}}\mathbf{Z}$ carries a canonical Lie algebroid structure like every Atiah bundle of a principal bundle. The Lie bracket is inherited from $\mathcal{D}^1(\mathbf{Z})$. The bracket can be also described in terms of affine derivations:

$$[X, X'] = X_{\mathsf{V}} \circ X' - (X')_{\mathsf{V}} \circ X,$$

where X_{V} is the vector part of the affine derivation $X : \mathsf{Sec}(\mathbf{Z}) \to C^\infty(M)$ (which represents also the anchor of X).

Similarly, the bundle $\tilde{\mathsf{L}}\mathbf{Z}$ is also canonically a Lie algebroid with similarly defined bracket

$$[D, D'] = D_{\mathsf{V}} \circ D' - (D')_{\mathsf{V}} \circ D,$$

where $D_{\mathsf{V}} : C^\infty(M) \to C^\infty(M)$ is the vector part of $D \in \mathsf{ADO}^1(\mathbf{Z})$.

Recall that the distinguished sections $X_{\mathsf{R}\mathbf{Z}} = -\partial_s$ and $I_{\mathsf{R}\mathbf{Z}} = 1$ are in the Lie algebroid $\mathsf{R}\mathbf{Z}$ *ideal sections*, i.e. these sections are nowhere-vanishing and the sections of the one-dimensional subbundles generated by $X_{\mathsf{R}\mathbf{Z}}$ and $I_{\mathsf{R}\mathbf{Z}}$ are Lie ideals with respect to the Lie algebroid bracket. The closed 'one-form' corresponding to $X_{\mathsf{R}\mathbf{Z}}$ we denote by ϕ^0.

The special affine bundles $\bar{\mathsf{T}}\mathbf{Z}$ and $\bar{\mathsf{L}}\mathbf{Z}$ also carry canonical algebraic structures, represented by the commutators of their sections regarded as affine maps $D : \mathsf{Sec}(\mathbf{Z}) \to \mathsf{Sec}(\mathbf{Z})$:

$$[D, D'] = D \circ D' - D' \circ D.$$

These structures can be recognized as *Lie affgebroid* structures. Recall (cf. [2]) that an *affine Lie bracket* on an affine space $\mathcal{A}$ is a bi-affine map

$$[\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \to \mathsf{V}(\mathcal{A})$$

which is skew-symmetric: $[\sigma_1, \sigma_2] = -[\sigma_2, \sigma_1]$ and satisfies the Jacobi identity:

$$[\sigma_1, [\sigma_2, \sigma_3]]^2_{\mathsf{V}} + [\sigma_2, [\sigma_3, \sigma_1]]^2_{\mathsf{V}} + [\sigma_3, [\sigma_1, \sigma_2]]^2_{\mathsf{V}} = 0,$$

where $[\cdot, \cdot]^2_{\mathsf{V}}$ is the affine-linear part of the biaffine bracket. An affine space equipped with an affine Lie bracket we shall call a *Lie affgebra*. Note that the term *affine Lie algebra* has been already used for certain types of Kac–Moody algebras.

If A is an affine bundle over M modeled on $\mathsf{V}(A)$ then a *Lie affgebroid structure* on A is an affine Lie bracket on sections of A and a morphism $\gamma : A \to \mathsf{T}M$ of affine bundles (over the identity on M) such that $[\sigma, \cdot]^2_{\mathsf{V}}$ is a quasi-derivation with the anchor $\gamma(\sigma)$, i.e.

$$[\sigma, fX]^2_{\mathsf{V}} = f[\sigma, X]^2_{\mathsf{V}} + \gamma(\sigma)(f)X$$

for all $\sigma \in \mathsf{Sec}(A)$, $X \in \mathsf{Sec}(\mathsf{V}(A))$, $f \in C^\infty(M)$.

Remark. The above definition is a slight generalization of the one proposed in [12,13] where the additional assumptions that the base manifold M is fibered over $\mathbb{R}$ and that $\gamma(\sigma)$ are vector fields projectable onto $\partial/\partial t$ have been put. On the other hand, one can try to define Lie affgebra as a skew-symmetric (in the affine sense) bracket $[\cdot,\cdot]_a$ on $\mathsf{Sec}(A)$ with values in $\mathsf{Sec}(A)$ satisfying the Jacobi identity of the form

$$\sum_{\omega\in S_3}\mathrm{sgn}(\omega)[\sigma_{\omega(1)},[\sigma_{\omega(2)},\sigma_{\omega(3)}]_a]_a=0.$$

The l.h.s. of the above equation is a vector combination of elements of $\mathsf{Sec}(A)$, so the identity makes sense. The problem with such definition is that, as we already know, any skew-symmetric operation on $\mathsf{Sec}(A)$ defines automatically an element $\sigma_0 \in \mathsf{Sec}(A)$, $\sigma_0=[\sigma,\sigma]$, and we get such a bracket in the form $[\sigma_1,\sigma_2]_a=[\sigma_1,\sigma_2]+\sigma_0$, where $[\cdot,\cdot]$ is the Lie affgebra bracket in the version we started with. Fixing σ_0 is usually too much (we just get a trivialization of the affine space) for applications and canonical examples, so we remain with the weaker definition.

Example 1. Every AV-bundle $\mathbf{Z}$ carries a canonical Lie affgebroid structure induced by the affine structure. The bracket of sections σ,σ' of $\mathbf{Z}$ is just $[\sigma,\sigma']=(\sigma-\sigma')$.

The following fact has been proved in [2], Theorem 11.

Theorem 19. *A map* $[\cdot,\cdot]:\mathsf{Sec}(A)\times\mathsf{Sec}(A)\to\mathsf{Sec}(\mathsf{V}(A))$ *is a Lie affgebroid bracket on an affine bundle A if and only if there is an extension of this map to a Lie algebroid bracket* $[\cdot,\cdot]^\wedge$ *on* $\hat{A}$ *such that*

$$[\mathsf{Sec}(\hat{A}),\mathsf{Sec}(\hat{A})]^\wedge\subset\mathsf{Sec}(\mathsf{V}(A)). \tag{33}$$

Moreover, (33) *is equivalent to the fact that* $1_A\in\mathsf{Sec}(A^\dagger)=\mathsf{Sec}(\hat{A}^*)$ *is a closed one-form.*

The Lie algebroid $(\hat{A},[\cdot,\cdot]^\wedge)$ is uniquely determined by the Lie affgebroid $(A,[\cdot,\cdot])$ and we will call it the *Lie algebroid hull* of $(A,[\cdot,\cdot])$.

Example 2. The Lie affgebroid bracket on $\mathbf{Z}$ from the previous example extends to a Lie algebroid bracket on $\hat{\mathbf{Z}}$. This bracket can be expressed by means of the canonical Jacobi bracket on $\mathbf{Z}$ by $\mathbf{F}_{[u,u']}=\{\mathbf{F}_u,\mathbf{F}_{u'}\}_{\mathbf{Z}}$.

Since the affine subbundles $\bar{\mathsf{T}}\mathbf{Z}$ and $\bar{\mathsf{L}}\mathbf{Z}$ in the Lie algebroids $\check{\mathsf{L}}\mathbf{Z}$ and $\tilde{\mathsf{R}}\mathbf{Z}$ are defined as the 1-level sets of ϕ^0 and $\phi^1-\phi^0$, respectively, we get the following theorem (cf. [2]).

Theorem 20. *The special affine bundles* $\bar{\mathsf{T}}\mathbf{Z}$ *and* $\bar{\mathsf{L}}\mathbf{Z}$ *carry canonical Lie affgebroid structures for which the brackets are the commutators in* $\mathsf{ADer}(\mathbf{Z};\mathbf{Z})$ *and* $\mathsf{ADO}^1(\mathbf{Z};\mathbf{Z})$, *respectively. The Lie affgebroid hulls of* $\bar{\mathsf{T}}\mathbf{Z}$ *and* $\bar{\mathsf{L}}\mathbf{Z}$ *are* $\check{\mathsf{L}}\mathbf{Z}$ *and* $\tilde{\mathsf{R}}\mathbf{Z}$, *respectively.*

12. Aff-Poisson and aff-Jacobi brackets

The idea of an affine analog of a Poisson bracket goes back to [24] but we will mainly follow the picture described in [2].

Let $\mathbf{Z}$ be an AV-bundle over M. An affine Lie bracket on $\mathsf{Sec}(\mathbf{Z})$

$$\{\cdot,\cdot\} : \mathsf{Sec}(\mathbf{Z}) \times \mathsf{Sec}(\mathbf{Z}) \to C^{\infty}(M)$$

is called an *aff-Poisson* (resp.*aff-Jacobi*) *bracket* if

$$\{\sigma,\cdot\} : \mathsf{Sec}(\mathbf{Z}) \to C^{\infty}(M)$$

is an affine derivation (resp. an affine first-order differential operator) for every $\sigma \in \mathsf{Sec}(\mathbf{Z})$.

We use the term *aff-Poisson*, since *affine Poisson structure* has already a different meaning in the literature.

Example 3. Every AV-bundle $\mathbf{Z}$ carries a canonical aff-Jacobi bracket determined by the affine structure:

$$\{\sigma,\sigma'\} = \sigma - \sigma'. \tag{34}$$

Theorem 21 (Grabowska et al. [2]). *For every aff-Poisson* (*resp. aff-Jacobi*) *bracket*

$$\{\cdot,\cdot\} : \mathsf{Sec}(\mathbf{Z}) \times \mathsf{Sec}(\mathbf{Z}) \to C^{\infty}(M)$$

its vector part

$$\{\cdot,\cdot\}_{\mathsf{V}} : C^{\infty}(M) \times C^{\infty}(M) \to C^{\infty}(M)$$

is a Poisson (*resp. Jacobi*) *bracket. Moreover,*

$$\{\sigma,\cdot\}^2_{\mathsf{V}} : C^{\infty}(M) \to C^{\infty}(M)$$

is a derivation (*resp. first-order differential operator*) *for every section* $\sigma \in \mathsf{Sec}(\mathbf{Z})$, *which is simultaneously a derivation of the bracket* $\{\cdot,\cdot\}_{\mathsf{V}}$. *Conversely, if we have a Poisson* (*resp. Jacobi*) *bracket* $\{\cdot,\cdot\}_0$ *on* $C^{\infty}(M)$ *and a derivation* (*resp. a first-order differential operator*)

$$D : C^{\infty}(M) \to C^{\infty}(M)$$

which is simultaneously a derivation of the bracket $\{\cdot,\cdot\}_0$, *then there is a unique aff-Poisson* (*resp. aff-Jacobi*) *bracket* $\{\cdot,\cdot\}$ *on* $\mathsf{Sec}(\mathbf{Z})$ *such that* $\{\cdot,\cdot\}_0 = \{\cdot,\cdot\}_{\mathsf{V}}$ *and* $D = \{\sigma,\cdot\}^2_{\mathsf{V}}$ *for a chosen section* $\sigma \in \mathsf{Sec}(\mathbf{Z})$.

Using a section σ_0 *to identify* $\mathsf{Sec}(\mathbf{Z})$ *with* $C^{\infty}(M)$, *we get that the aff-Poisson* (*resp. aff-Jacobi*) *bracket on* $\mathsf{Sec}(\mathbf{Z})$ *has the form*

$$\{\sigma,\sigma'\} = D(\sigma' - \sigma) + \{\sigma,\sigma'\}_{\mathsf{V}},$$

where D *is a vector field* (*resp first-order differential operator*) *which is a derivation of the Poisson* (*resp. Jacobi*) *bracket* $\{\cdot,\cdot\}_{\mathsf{V}}$.

Example 4. Every Poisson (resp., Jacobi) bracket $\{\cdot,\cdot\}_M$ on $C^{\infty}(M)$ can be interpreted as an aff-Poisson (resp., aff-Jacobi) bracket $\{\cdot,\cdot\}$ on sections of the trivial AV-affine bundle $M \times \mathbf{I}$. In this case the trivialization is canonical, $D = 0$ and $\{\cdot,\cdot\} = \{\cdot,\cdot\}_M$.

Theorem 22. *Let $\zeta : \mathbf{Z} \to M$ be an AV-bundle. Then*:

(1) *There is a one-to-one correspondence between aff-Poisson brackets $\{\cdot,\cdot\}_{aP}$ on $\mathsf{Sec}(\mathbf{Z})$ and Poisson brackets $\{\cdot,\cdot\}_{\Pi}$ on $C^\infty(\mathbf{Z})$ which are $X_{\mathbf{Z}}$-invariant, i.e. which are associated with Poisson tensors Π on $\mathbf{Z}$ such that $\pounds_{X_{\mathbf{Z}}}\Pi = 0$. This correspondence is determined by*

$$\{\sigma,\sigma'\}_{aP}\circ\zeta = \{\mathbf{F}_\sigma,\mathbf{F}_{\sigma'}\}_{\Pi}. \tag{35}$$

(2) *There is a one-to-one correspondence between aff-Jacobi brackets $\{\cdot,\cdot\}_{aJ}$ on $\mathsf{Sec}(\mathbf{Z})$ and Jacobi brackets $\{\cdot,\cdot\}_J$* on $C^\infty(\mathbf{Z})$ which are associated with Jacobi structures $J = (\Pi,\Gamma)$ on $\mathbf{Z}$ such that $\pounds_{X_{\mathbf{Z}}}\Gamma = 0$ and $\pounds_{X_{\mathbf{Z}}}\Pi = \Gamma\wedge X_{\mathbf{Z}}$. This correspondence is determined by

$$\{\sigma,\sigma'\}_{aJ}\circ\zeta = \{\mathbf{F}_\sigma,\mathbf{F}_{\sigma'}\}_J. \tag{36}$$

Proof. We will prove only part (2). The proof of (1) is analogous but easier. Since all objects are local over M, we can use local affine coordinates (x^a, s) on $\mathbf{Z}$ in which $X_{\mathbf{Z}} = -\partial_s$ and identify sections σ of $\mathbf{Z}$ with functions $\sigma(x)$, so that $\mathbf{F}_\sigma(x,s) = \sigma(x) - s$. We will identify functions on M with basic functions on $\mathbf{Z}$. Assume first that $\{\cdot,\cdot\}_{aJ}$ is an aff-Jacobi bracket on $\mathsf{Sec}(\mathbf{Z})$. According to Theorem 21 there is a Jacobi structure $J_0 = (\Pi_0,\Gamma_0)$ on M and a first-order differential operator $D = \mathring{D} + f$ on M such that $\{\sigma,\sigma'\}_{aJ} = D(\sigma-\sigma') + \{\sigma,\sigma'\}_{J_0}$. The equation (36) can be rewritten in the form

$$\begin{aligned}&\Pi_0(\sigma,\sigma') + \sigma\Gamma_0(\sigma') - \sigma'\Gamma_0(\sigma) + \mathring{D}(\sigma-\sigma') + f(\sigma-\sigma')\\ &\quad = \Pi(\sigma-s,\sigma'-s) + (\sigma-s)\Gamma(\sigma'-s) - (\sigma'-s)\Gamma(\sigma-s),\end{aligned}$$

which has a unique solution $J = (\Pi,\Gamma)$, namely

$$\Pi = \Pi_0 + \partial_s\wedge(\mathring{D} - s\Gamma_0), \qquad \Gamma = \Gamma_0 - f\partial_s.$$

It is easy to see that the Jacobi identity for $\{\cdot,\cdot\}_{aJ}$ implies the Jacobi identity for $\{\cdot,\cdot\}_J$ and that this solution has the required properties with respect to $\pounds_{X_{\mathbf{Z}}}$.

Conversely, assume that J is a Jacobi structure on $\mathbf{Z}$. The conditions $\pounds_{X_{\mathbf{Z}}}\Gamma = 0$ and $\pounds_{X_{\mathbf{Z}}}\Pi = \Gamma\wedge X_{\mathbf{Z}}$ imply that $\pounds_{X_{\mathbf{Z}}}(\{(\sigma-s),(\sigma'-s)\}_J) = 0$, i.e. $\{(\sigma-s),(\sigma'-s)\}_J$ is a basic function, so that (21) defines a bracket on $\mathsf{Sec}(\mathbf{Z})$. It is easy to see that this bracket is an aff-Jacobi bracket. □

Note, that every skew-symmetric affine bracket $\{\cdot,\cdot\}$ is uniquely determined by $\{\sigma,\cdot\}^2_{\mathsf{V}}$, namely

$$\{\sigma,\sigma'\} = \{\sigma,\sigma'-\sigma\}^2_{\mathsf{V}}. \tag{37}$$

For an aff-Poisson bracket on sections of $\mathbf{Z}$ the mapping $f \mapsto \{\sigma, f\}^2_{\mathsf{V}}$ is a derivation of the algebra $C^\infty(M)$, hence a vector field on M. We denote it by X_σ and we call it the *Hamiltonian vector field* of σ.

Example 5. The canonical Poisson structure Π on the cotangent bundle T^*M is invariant with respect to translation by the vertical lift α_{T^*M} of any closed one-form $\alpha\in\mathsf{Sec}(\mathsf{T}^*M)$. If α is nowhere-vanishing, we can consider the corresponding AV-bundle $\mathbf{Z} = \mathsf{AV}(\mathsf{T}^*M)$ for which $X_{\mathbf{Z}}$ is the vertical lift of α, i.e. $X_{\mathbf{Z}} = \alpha_{\mathsf{T}^*M}$. Hence, the AV-bundle $\mathbf{Z}$, i.e. $\zeta : \mathsf{T}^*M \to \mathsf{T}^*M/\langle\alpha\rangle$, carries a canonical aff-Poisson structure with the bracket (35). Since, for any section σ of $\mathbf{Z}$ and for any function f on $\mathsf{T}^*M/\langle\alpha\rangle$, we have $(\{\sigma, f\}_{aP})^2_{\mathsf{V}} \circ \zeta = \{\mathbf{F}_\sigma, f \circ \zeta\}_\Pi$, the Hamiltonian vector field X_σ on $\mathsf{T}^*M/\langle\alpha\rangle$ induced by the section σ is the projection of the Hamiltonian vector field on T^*M induced by $\mathbf{F}_\sigma$.

In the theory of Lie algebroids it is well-known that a Lie algebroid brackets $[\cdot, \cdot]$ on the vector bundle E are in a one-to-one correspondence with linear Poisson brackets $\{\cdot, \cdot\}$ on E^*. Linearity of the bracket means that the bracket of linear functions is a linear function and the correspondence is described by

$$\{\iota_{E^*}(X_1), \iota_{E^*}(X_2)\} = \iota_{E^*}([X_1, X_2]),$$

where $\iota_{E^*}(X)$ denotes the linear function on E^* associated canonically with $X \in \mathsf{Sec}(E)$. In [3] it has been shown that this correspondence can be extended to a one-to-one correspondence between Lie algebroid brackets on E and affine Jacobi brackets (bracket of affine functions is an affine function) on an arbitrary affine hyperbundle A of E^*. In the case of special affine bundles we have an analogous correspondence which refers to Theorem 13.

Let $\mathbf{A} = (A, v^0)$ be a special affine bundle over M. There is an obvious identification of a section X of $\mathsf{V}(\mathbf{A})$ with a linear function $\iota_{\mathbf{A}^\dagger}(X)$ on $\mathbf{A}^\dagger$ and an affine function $\iota_{\mathbf{A}^\#}(X)$ on $\mathbf{A}^\#$ which are invariant with respect to translation by $1_{\mathbf{A}}$, so they are pull-backs of a certain linear function $\iota^\dagger_X$ and an affine function $\iota^\#_X$ on $\mathbf{A}^\dagger/\langle 1_{\mathbf{A}}\rangle$ and $\mathbf{A}^\#/\langle 1_{\mathbf{A}}\rangle$, respectively.

Theorem 23 (Grabowska et al. [2]). *There is a one-to-one correspondence between Lie affgebroid brackets* $[\cdot, \cdot]_{\mathbf{A}}$ *on a special affine bundle* $\mathbf{A}$ *and*

(1) *linear aff-Poisson brackets* $\{\cdot, \cdot\}_{\mathbf{A}^\dagger}$ *on the AV-bundle* $\mathsf{AV}(\mathbf{A}^\dagger)$, *i.e. on* $\rho^\dagger : \mathbf{A}^\dagger \to \mathbf{A}^\dagger/\langle 1_{\mathbf{A}}\rangle$, *determined by*

$$\iota^\dagger_{[a,a']_{\mathbf{A}}} = \{\hat{\sigma}_a, \hat{\sigma}_{a'}\}_{\mathbf{A}^\dagger}, \tag{38}$$

(2) *affine aff-Jacobi brackets* $\{\cdot, \cdot\}_{\mathbf{A}^\#}$ *on the AV-bundle* $\mathsf{AV}(\mathbf{A}^\#)$, *i.e. on* $\rho : \mathbf{A}^\# \to \mathbf{A}^\#/\langle 1_{\mathbf{A}}\rangle$, *determined by*

$$\iota^\#_{[a,a']_{\mathbf{A}}} = \{\sigma_a, \sigma_{a'}\}_{\mathbf{A}^\#}. \tag{39}$$

This aff-Jacobi bracket is aff-Poisson if and only if the section v_0 *is central in the Lie algebroid hull* $\hat{\mathbf{A}}$ *of* $\mathbf{A}$.

Remark. Here we call an aff-Poisson (resp., aff-Jacobi) structure linear (resp., affine) if the bracket of linear sections of $\rho^\dagger : \mathbf{A}^\dagger \to \mathbf{A}^\dagger/\langle 1_{\mathbf{A}}\rangle$ (resp., the bracket of affine sections of $\rho : \mathbf{A}^\# \to \mathbf{A}^\#/\langle 1_{\mathbf{A}}\rangle$) is a linear function on $\mathbf{A}^\dagger/\langle 1_{\mathbf{A}}\rangle$ (resp., it is an affine function on $\mathbf{A}^\#/\langle 1_{\mathbf{A}}\rangle$).

Proof. Part (1) has been proved in [2], Theorem 19, so $[\cdot, \cdot]_{\mathbf{A}}$ induces certain aff-Poisson bracket $\{\cdot, \cdot\}_{\mathbf{A}^\dagger}$. According to Theorem 22, there is a Poisson tensor Π on $\mathbf{A}^\dagger$ which corresponds to the aff-Poisson bracket $\{\cdot, \cdot\}_{\mathbf{A}^\dagger}$. This tensor is, clearly, linear and it is invariant with respect to the vector field $X^\dagger = (1_{\mathbf{A}})_{\mathbf{A}^\dagger}$—the vertical lift of the section $1_{\mathbf{A}}$. Now, we use the result [3], Corollary 3.6, which implies that there is a one-to-one correspondence between linear Poisson brackets $\{\cdot,\cdot\}_\Pi$ on $\mathbf{A}^\dagger$ and affine Jacobi brackets $\{\cdot,\cdot\}_J$ on $\mathbf{A}^\#$ such that

$$\{u, v\}_\Pi|_{\mathbf{A}^\#} = \{u|_{\mathbf{A}^\#}, v|_{\mathbf{A}^\#}\}_J$$

for all linear functions u, v on $A^\dagger$. The Jacobi structure $J = (\Pi_0, \Gamma_0)$ is the restriction to $\mathbf{A}^\#$ of the Jacobi structure $(\Pi + \Gamma \wedge \Delta_{\mathbf{A}^\dagger}, \Gamma)$, where $\Gamma = \{v^0, \cdot\}_\Pi$ is the Hamiltonian vector field of the linear function $\iota_{\mathbf{A}^\dagger}(v^0) \in \mathsf{Sec}(\mathsf{V}(\mathbf{A}))$ which defines the affine hyperbundle $\mathbf{A}^\#$ in $\mathbf{A}^\dagger$ and $\Delta_{\mathbf{A}^\dagger}$ is the Liouville (Euler) vector field on the vector bundle $\mathbf{A}^\dagger$. The crucial point is that $X^\dagger$ preserves Π if and only if it preserves Γ and

$$\pounds_{X^\dagger}(\Pi + \Gamma \wedge \Delta_{\mathbf{A}^\dagger}) = \Gamma \wedge X^\dagger. \tag{40}$$

Indeed, since the vector field $X^\dagger$ preserves Π, and the function $\iota_{\mathbf{A}^\dagger}(v^0)$ due to the fact that $X^\dagger(\iota_{\mathbf{A}^\dagger}(v^0))$ is the pull-back of $\langle 1_{\mathbf{A}}, v^0\rangle = 0$, it preserves also Γ, i.e. $\pounds_{X^\dagger}\Gamma = 0$. Moreover, since $X^\dagger$ is a vertical lift, $\pounds_{X^\dagger}\Delta_{\mathbf{A}^\dagger} = X^\dagger$. Thus we get (40) and, due to Theorem 23(b), we get an aff-Jacobi bracket $\{\cdot,\cdot\}_{\mathbf{A}^\#}$ on $\mathsf{AV}(\mathbf{A}^\#)$. It is easy to see that it satisfies (39). The converse is proved by a similar reasoning in reversed order. Passing to the restrictions to $\mathbf{A}^\#$ we get, in view of (22), that J corresponds to an aff-Jacobi bracket on sections of ρ. Since $\mathbf{F}^\dagger_{\hat\sigma_a}|_{\mathbf{A}^\#} = \mathbf{F}^\#_{\sigma_a}$, the theorem is proved. □

Example 6. The canonical Lie affgebroid structure on $\mathbf{Z}$ given by $[\sigma, \sigma'] = (\sigma' - \sigma)$ induces an aff-Poisson structure on the AV-bundle $\mathsf{AV}(\mathbf{Z}^\dagger)$ and an aff-Jacobi structure on $\mathsf{AV}(\mathbf{Z}^\#) = \mathbf{Z}$. The corresponding linear Poisson structure on $\mathbf{Z}^\dagger$ (resp., affine Jacobi structure on $\mathbf{Z}$) is $\Pi = \Delta_{\mathbf{Z}^\dagger} \wedge X_{\mathbf{Z}}$ (resp., $J = (0, X_{\mathbf{Z}})$).

Example 7. Consider the Lie algebroid structure on $\tilde{\mathsf{T}}\mathbf{Z}$ as a Lie affgebroid structure on the special affine bundle (in fact, special vector bundle). The special affine dual $(\tilde{\mathsf{T}}\mathbf{Z})^\#$ is $\mathsf{P}\mathbf{Z} \times^a \mathbf{I}$, so $\mathsf{AV}((\tilde{\mathsf{T}}\mathbf{Z})^\#)$ is the trivial AV-bundle over $\mathsf{P}\mathbf{Z}$. The corresponding aff-Jacobi bracket on $\mathsf{P}\mathbf{Z} \times^a \mathbf{I}$ is the aff-Poisson bracket induced from the canonical Poisson structure on $\mathsf{P}\mathbf{Z}$ associated with the canonical symplectic structure.

Example 8. Recall that we have the identification $\bar{\mathsf{T}}\mathbf{Z} = \tilde{\mathsf{T}}\mathbf{Z} \boxtimes_M \mathbf{Z} = (\mathsf{P}\mathbf{Z})^\dagger \boxtimes_M \mathbf{Z}$, so that, according to Theorem 5, $(\bar{\mathsf{T}}\mathbf{Z})^\# = \mathsf{P}\mathbf{Z} \times^a{}_M \mathbf{Z}^\#$. The corresponding AV-bundle is $\rho : \mathsf{P}\mathbf{Z} \times^a{}_M \bar{\mathbf{Z}}^\# \to \mathsf{P}\mathbf{Z}$. But $\mathsf{P}\mathbf{Z} \times^a{}_M \bar{\mathbf{Z}}^\# = \mathsf{P}\mathbf{Z} \times^a{}_M \mathbf{Z} = \mathsf{C}\mathbf{Z}$, so that $\mathsf{AV}((\bar{\mathsf{T}}\mathbf{Z})^\#) = \mathsf{AV}(\mathsf{C}\mathbf{Z})$. The special affine bundle $\bar{\mathsf{T}}\mathbf{Z}$ is canonically a Lie affgebroid, so, due to the above theorem, $\mathsf{AV}(\mathsf{C}\mathbf{Z})$ is equipped with a canonical aff-Jacobi structure. It is easy to guess that this is the structure corresponding, *via* Theorem 22, to the canonical Jacobi bracket (27) on $\mathsf{C}\mathbf{Z}$ associated with the canonical contact form which, in turn, is represented by the affine Liouville one-form. Indeed, let (x^a, p_b, s) be standard affine coordinates on $\mathsf{C}\mathbf{Z}$ induced from the Darboux coordinates in $\mathsf{T}^*M \times \mathbb{R}$ and let (x^a, f_b, β) be the coordinates in $\bar{\mathsf{T}}\mathbf{Z}$ representing the vector field

$$X = f_a(x)\partial_{x^a} + (s - \beta(x))\partial_s \in \mathcal{X}_{ah}(\mathbf{Z}).$$

The duality between $\bar{\mathsf{T}}\mathbf{Z} = (\mathsf{P}\mathbf{Z})^\dagger \boxtimes_M \mathbf{Z}$ and $\mathsf{C}\mathbf{Z} = \mathsf{P}\mathbf{Z}\times^a{}_M\mathbf{Z}$ is given by $\langle(x^a, f_b, \beta), (x^a, p_b, s)\rangle_{as} = f_a p_a + \beta - s$ so that $\iota^\#_{\mathsf{C}\mathbf{Z}}(X)(x^a, p_b, s) = f_a(x)p_a + \beta(x) - s$ and $\sigma_X(x^a, p_b) = (x^a, p_b, f_a(x)p_a + \beta(x))$. Since the Lie affgebroid bracket in $\bar{\mathsf{T}}\mathbf{Z}$ reads

$$\begin{aligned}
[X, X']_{\bar{\mathsf{T}}\mathbf{Z}} &= [f_a(x)\partial_{x^a} + (s-\beta(x))\partial_s, f'_b(x)\partial_{x^b} + (s-\beta'(x))\partial_s]_{\mathcal{D}^1(\mathbf{Z})}\\
&= \left(f_b(x)\frac{\partial f'_a}{\partial x^b}(x) - f'_b(x)\frac{\partial f_a}{\partial x^b}(x)\right)\partial_{x^a}\\
&\quad - \left(f_a(x)\frac{\partial \beta'}{\partial x^a}(x) - f'_a(x)\frac{\partial \beta}{\partial x^a}(x) + \beta(x) - \beta'(x)\right)\partial_s,
\end{aligned}$$

the corresponding Jacobi bracket on $\mathsf{C}\mathbf{Z}$ is uniquely characterized by

$$\begin{aligned}
&\{f_a(x)p_a + \beta(x) - s, f'_b(x)p_b + \beta'(x) - s\}_J\\
&\quad = p_a\left(f_b(x)\frac{\partial f'_a}{\partial x^b}(x) - f'_b(x)\frac{\partial f_a}{\partial x^b}(x)\right)\\
&\qquad + \left(f_a(x)\frac{\partial \beta'}{\partial x^a}(x) - f'_a(x)\frac{\partial \beta}{\partial x^a}(x) + \beta(x) - \beta'(x)\right).
\end{aligned}$$

It is easy to check that this is exactly the Jacobi structure

$$J = (\partial_{p_a}\wedge\partial_{x^a} + p_a\partial_{p_a}\wedge\partial_s, -\partial_s),$$

i.e. the Jacobi structure of the contact one-form $p_a\,\mathrm{d}x^a - \mathrm{d}s$.

13. Aff-Poisson and aff-Jacobi (co)homology

Let $\mathbf{Z}$ be an AV-bundle over M. It is obvious that affine biderivations on $\mathbf{Z}$ are affine derivations on $\mathbf{Z}$ with values in $\mathsf{ADer}(\mathbf{Z})$, i.e. sections of the bundle

$$\mathsf{Aff}_M(\mathsf{P}\mathbf{Z}; \tilde{\mathsf{T}}\mathbf{Z}) = \mathsf{Hom}_M(\widehat{\mathsf{P}\mathbf{Z}}; \tilde{\mathsf{T}}\mathbf{Z})) = (\mathsf{P}\mathbf{Z})^\dagger \otimes_M \tilde{\mathsf{T}}\mathbf{Z} = \tilde{\mathsf{T}}\mathbf{Z}\otimes_M\tilde{\mathsf{T}}\mathbf{Z}.$$

In this picture, skew-symmetric affine biderivations are sections of $\wedge^2\tilde{\mathsf{T}}\mathbf{Z}$. Similarly, affine first-order bidifferential operators on $\mathbf{Z}$ are sections of $\wedge^2\tilde{\mathsf{L}}\mathbf{Z}$. Since both, $\tilde{\mathsf{T}}\mathbf{Z}$ and $\tilde{\mathsf{L}}\mathbf{Z}$, are Lie algebroids, there are the corresponding Lie algebroid Schouten brackets $[\![\cdot,\cdot]\!]_{\tilde{\mathsf{T}}\mathbf{Z}}$ and $[\![\cdot,\cdot]\!]_{\tilde{\mathsf{L}}\mathbf{Z}}$ on the Grassmann algebras $\mathsf{A}(\tilde{\mathsf{T}}\mathbf{Z}) = \oplus_n\mathsf{A}^n(\tilde{\mathsf{T}}\mathbf{Z}) = \oplus_n\mathsf{Sec}(\wedge^n\tilde{\mathsf{T}}\mathbf{Z})$ and $\mathsf{A}(\tilde{\mathsf{L}}\mathbf{Z}) = \oplus_n\mathsf{Sec}(\wedge^n\tilde{\mathsf{L}}\mathbf{Z})$ of the vector bundles $\tilde{\mathsf{T}}\mathbf{Z}$ and $\tilde{\mathsf{L}}\mathbf{Z}$, respectively. What will be crucial here is that the Lie algebroid $\tilde{\mathsf{L}}\mathbf{Z}$ possesses additionally a canonical closed 'one-form' ϕ^0 inherited from $\mathsf{R}\mathbf{Z}$ (in fact, $\phi^0 = \phi^1$ on $\tilde{\mathsf{L}}\mathbf{Z}$) which makes it into a *Jacobi algebroid* with the Schouten–Jacobi bracket $[\![\cdot,\cdot]\!]^{\phi^0}_{\tilde{\mathsf{L}}\mathbf{Z}}$.

Remark. The Jacobi algebroids have been introduced by Iglesias and Marrero [10] under the name *generalized Lie algebroids* and recognized as graded Jacobi brackets in [4,5]. For the definitions and details we refer to these papers or to the article [9] which contains a brief introduction to the theory of Jacobi algebroids, the corresponding lifts of tensors and *canonical structures*.

Theorem 24.

(a) $\Lambda \in \mathsf{Sec}(\wedge^2 \tilde{\mathsf{T}}\mathbf{Z})$ *represents an aff-Poisson structure on* $\mathbf{Z}$ *if and only if* $[\![\Lambda, \Lambda]\!]_{\tilde{\mathsf{T}}\mathbf{Z}} = 0$.
(b) $\mathcal{J} \in \mathsf{Sec}(\wedge^2 \tilde{\mathsf{L}}\mathbf{Z})$ *represents an aff-Jacobi structure on* $\mathbf{Z}$ *if and only if* $[\![\mathcal{J}, \mathcal{J}]\!]^{\phi^0}_{\tilde{\mathsf{L}}\mathbf{Z}} = 0$.

In other words, aff-Poisson and aff-Jacobi structures are canonical structures for the Lie algebroid $\tilde{\mathsf{T}}\mathbf{Z}$ *and the Jacobi algebroid* $(\tilde{\mathsf{L}}\mathbf{Z}, \phi^0)$, *respectively*.

Proof.

(a) We will use a trivialization of $\mathbf{Z}$ to identify $\tilde{\mathsf{T}}\mathbf{Z}$ with $\mathsf{L}M = \mathsf{T}M \oplus \mathbb{R}$ and we will use the expression $D = (X, \beta) \in \mathsf{Sec}(\mathsf{L}M) = \mathcal{X}(M) \times C^\infty(M)$ for sections D of $\tilde{\mathsf{T}}\mathbf{Z}$ (see the convention preceding (30)). The action on $\sigma \in \mathsf{Sec}(\mathbf{Z})$, identified with functions on M, reads $D(\sigma) = X(\sigma) + \beta$. With respect to this identification, sections of $\tilde{\mathsf{T}}\mathbf{Z}$ commute exactly as sections of $\mathsf{L}M$, i.e. (cf. (30))

$$[(X, \beta), (X', \beta')]_{\tilde{\mathsf{T}}\mathbf{Z}} = ([X, X']_{\mathsf{T}M}, X(\beta') - X'(\beta)).$$

Since $\wedge^2 \tilde{\mathsf{T}}\mathbf{Z}$ is identified with $\wedge^2(\mathsf{T}M \oplus \mathbb{R})$, elements $\Lambda \in \mathsf{Sec}(\wedge^2 \tilde{\mathsf{T}}\mathbf{Z})$ are of the form $\Lambda = \Lambda_0 + X_{\mathbf{Z}} \wedge X_0$, where $X_{\mathbf{Z}} = (0, 1)$, $\Lambda_0 \in \mathsf{Sec}(\wedge^2 \mathsf{T}M)$ is a bivector field on M and X_0 is a vector field on M. The bi-section Λ induces the bracket

$$\{\sigma, \sigma'\}_\Lambda = \{\sigma, \sigma'\}_{\Lambda_0} + X_0(\sigma' - \sigma).$$

In view of Theorem 21 this is an aff-Poisson bracket if and only if Λ_0 is a Poisson tensor and $[\![X_0, \Lambda_0]\!]^{SN} = 0$, where $[\![\cdot, \cdot]\!]^{SN}$ is the Schouten–Nijenhuis bracket, i.e. the Lie algebroid Schouten bracket for $\mathsf{T}M$. But these conditions are equivalent to $[\![\Lambda, \Lambda]\!]_{\tilde{\mathsf{T}}\mathbf{Z}} = 0$. Indeed, since $X_{\mathbf{Z}}$ is a central section,

$$\begin{aligned}[\![\Lambda, \Lambda]\!]_{\tilde{\mathsf{T}}\mathbf{Z}} &= [\![\Lambda_0, \Lambda_0]\!]_{\tilde{\mathsf{T}}\mathbf{Z}} + 2[\![\Lambda_0, X_{\mathbf{Z}} \wedge X_0]\!]_{\tilde{\mathsf{T}}\mathbf{Z}} + [\![X_{\mathbf{Z}} \wedge X_0, X_{\mathbf{Z}} \wedge X_0]\!]_{\tilde{\mathsf{T}}\mathbf{Z}} \\ &= [\![\Lambda_0, \Lambda_0]\!]^{SN} - 2X_{\mathbf{Z}} \wedge [\![\Lambda_0, X_0]\!]^{SN},\end{aligned}$$

that vanishes exactly when $[\![\Lambda_0, \Lambda_0]\!]^{SN} = 0$ and $[\![\Lambda_0, X_0]\!]^{SN} = 0$.

(b) Similarly as above we use an identification $\tilde{\mathsf{L}}\mathbf{Z} \simeq \mathsf{L} \oplus \mathbb{R}$ and the expression $D = (X, \beta) \in \mathsf{Sec}(\mathsf{L}M \oplus_M \mathbb{R}) = \mathcal{D}^1(M) \times C^\infty(M)$ for sections D of $\tilde{\mathsf{L}}\mathbf{Z}$. The action on $\sigma \in \mathsf{Sec}(\mathbf{Z})$ reads $D(\sigma) = X(\sigma) + \beta$. With respect to this identification, sections of $\tilde{\mathsf{L}}\mathbf{Z}$ commute like

$$[(X, \beta), (X', \beta')]_{\tilde{\mathsf{L}}\mathbf{Z}} = ([X, X']_{\mathsf{L}M}, X(\beta') - X'(\beta)).$$

Elements $\mathcal{J} \in \mathsf{Sec}(\wedge^2 \tilde{\mathsf{L}}\mathbf{Z})$ are of the form $\mathcal{J} = \mathcal{J}_0 + X_{\mathbf{Z}} \wedge D_0$, where $X_{\mathbf{Z}} = (0, 1)$, $\mathcal{J}_0 \in \mathsf{Sec}(\wedge^2 \mathsf{L}M)$ is a first-order bidifferential operator on M and D_0 is a first-order differential operator on M. The Schouten–Jacobi bracket $[\![\cdot, \cdot]\!]^{\phi^0}_{\tilde{\mathsf{L}}\mathbf{Z}}$ restricted to $\mathsf{L}M$ gives the canonical Schouten–Jacobi bracket $[\![\cdot,\cdot]\!]^{\phi^0}_{\mathsf{L}M}$ on $\mathsf{A}(\mathsf{L}M)$ for which canonical structures are Jacobi structures on M (cf. [4]). The section $\mathcal{J}$ induces the bracket

$$\{\sigma,\sigma'\}_{\mathcal{J}} = \{\sigma,\sigma'\}_{\mathcal{J}_0} + D_0(\sigma'-\sigma).$$

This is an aff-Jacobi bracket if and only if $\mathcal{J}_0$ is a Jacobi structure and $[\![D_0,\mathcal{J}_0]\!]^{\phi^0}_{\mathsf{L}M}=0$ (Theorem 21). But these conditions are equivalent to $[\![\mathcal{J},\mathcal{J}]\!]^{\phi^0}_{\tilde{\mathsf{L}}\mathbf{Z}}=0$. Indeed, since $X_{\mathbf{Z}}$ is an ideal section such that $[D,X_{\mathbf{Z}}]_{\tilde{\mathsf{L}}\mathbf{Z}}=\phi^0(D)X_{\mathbf{Z}}$, we have $[\![R,X_{\mathbf{Z}}]\!]^{\phi^0}_{\tilde{\mathsf{L}}\mathbf{Z}}=X_{\mathbf{Z}}\wedge i_{\phi^0}R$ for any $R\in\mathsf{Sec}(\mathsf{A}(\check{\mathsf{L}}\mathbf{Z}))$, and, due to the properties of the Schouten–Jacobi brackets,

$$\begin{aligned}[\![\mathcal{J},\mathcal{J}]\!]^{\phi^0}_{\tilde{\mathsf{L}}\mathbf{Z}} &= [\![\mathcal{J}_0,\mathcal{J}_0]\!]^{\phi^0}_{\tilde{\mathsf{L}}\mathbf{Z}} + 2[\![\mathcal{J}_0,X_{\mathbf{Z}}\wedge D_0]\!]^{\phi^0}_{\tilde{\mathsf{L}}\mathbf{Z}} + [\![X_{\mathbf{Z}}\wedge D_0,X_{\mathbf{Z}}\wedge D_0]\!]^{\phi^0}_{\tilde{\mathsf{L}}\mathbf{Z}}\\ &= [\![\mathcal{J}_0,\mathcal{J}_0]\!]^{\phi^0}_{\tilde{\mathsf{L}}\mathbf{Z}} + 2([\![\mathcal{J}_0,X_{\mathbf{Z}}]\!]^{\phi^0}_{\tilde{\mathsf{L}}\mathbf{Z}}\wedge D_0\\ &\quad -X_{\mathbf{Z}}\wedge[\![\mathcal{J}_0,D_0]\!]^{\phi^0}_{\tilde{\mathsf{L}}\mathbf{Z}} - i_{\phi^0}\mathcal{J}_0\wedge X_{\mathbf{Z}}\wedge D_0)\\ &= [\![\mathcal{J}_0,\mathcal{J}_0]\!]^{\phi^0}_{\mathsf{L}M} - 2X_{\mathbf{Z}}\wedge[\![\mathcal{J}_0,X_0]\!]^{\phi^0}_{\mathsf{L}M},\end{aligned}$$

that vanishes exactly when $[\![\mathcal{J}_0,\mathcal{J}_0]\!]^{\phi^0}_{\mathsf{L}M}=0$ and $[\![\mathcal{J}_0,X_0]\!]^{\phi^0}_{\mathsf{L}M}=0$, i.e. when $\mathcal{J}_0$ is a Jacobi structure for which D_0 acts as a derivation of the corresponding Jacobi bracket □

Since aff-Poisson and aff-Jacobi structures have been recognized as canonical structures, we can apply results of [9] to characterize them in terms of induced morphisms of vector bundles, and results of [4,5] to define the corresponding cohomology and homology.

For $Y\in\mathsf{Sec}(\mathsf{A}(\tilde{\mathsf{T}}\mathbf{Z}))$ denote by Y^c its Lie algebroid complete lift to a multivector field on $\tilde{\mathsf{T}}\mathbf{Z}$ (see [7,8] or the survey in [9]). Similarly, for $Y\in\mathsf{Sec}(\mathsf{A}(\tilde{\mathsf{L}}\mathbf{Z}))$ denote by $\hat{Y}_{\phi^0}$ its Jacobi algebroid complete lift to a first-order polydifferential operator on $\tilde{\mathsf{L}}\mathbf{Z}$ (see [4] or the survey in [9]). Let $\Lambda_{\tilde{\mathsf{T}}^*\mathbf{Z}}$ be the canonical linear Poisson structure on $\tilde{\mathsf{T}}^*\mathbf{Z}$ representing the Lie algebroid structure on $\tilde{\mathsf{T}}\mathbf{Z}$ and let $J_{\tilde{\mathsf{L}}^*\mathbf{Z}}$ be the canonical homogeneous Jacobi structure on the dual $\tilde{\mathsf{L}}^*\mathbf{Z}$ of $\tilde{\mathsf{L}}\mathbf{Z}$ representing the Jacobi algebroid structure on $\tilde{\mathsf{L}}\mathbf{Z}$.

Theorem 25.

(i) $\Lambda\in\mathsf{Sec}(\wedge^2\tilde{\mathsf{T}}\mathbf{Z})$ *represents an aff-Poisson structure on* $\mathbf{Z}$ *if and only if the tensors* $\Lambda_{\tilde{\mathsf{T}}^*\mathbf{Z}}$ *and* $-\Lambda^c$ *are* $\sharp_\Lambda$*-related, where* $\sharp_\Lambda:\tilde{\mathsf{T}}^*\mathbf{Z}\to\tilde{\mathsf{T}}\mathbf{Z}$, $\sharp_\Lambda(\mu_m)=i_{\mu_m}\Lambda(m)$.

(ii) $\mathcal{J}\in\mathsf{Sec}(\wedge^2\tilde{\mathsf{L}}\mathbf{Z})$ *represents an aff-Jacobi structure on* $\mathbf{Z}$ *if and only if the first-order bidifferential operators* $J_{\tilde{\mathsf{L}}^*\mathbf{Z}}$ *and* $-\hat{\mathcal{J}}_{\phi^0}$ *are* $\sharp_{\mathcal{J}}$*-related, where* $\sharp_{\mathcal{J}}:\tilde{\mathsf{L}}^*\mathbf{Z}\to\tilde{\mathsf{L}}\mathbf{Z}$, $\sharp_{\mathcal{J}}(\omega_m)=i_{\omega_m}\mathcal{J}(m)$.

Theorem 26.

(a) $\Lambda \in \mathsf{Sec}(\wedge^2 \tilde{\mathsf{T}}\mathbf{Z})$ *represents an aff-Poisson structure on* $\mathbf{Z}$ *if and only if the graded operator* $\partial_\Lambda(Y) = [\![\Lambda, Y]\!]_{\tilde{\mathsf{T}}\mathbf{Z}}$ *of degree* 1 *on* $\mathsf{A}(\tilde{\mathsf{T}}\mathbf{Z})$ *is a cohomology operator, i.e.* $(\partial_\Lambda)^2 = 0$.

(b) $\mathcal{J} \in \mathsf{Sec}(\wedge^2 \tilde{\mathsf{L}}\mathbf{Z})$ *represents an aff-Jacobi structure on* $\mathbf{Z}$ *if and only if the graded operators* $\partial^t_{\mathcal{J}}(R) = [\![\mathcal{J}, R]\!]^{\phi^0}_{\tilde{\mathsf{L}}\mathbf{Z}} + t i_{\phi^0}\mathcal{J} \wedge R$ *of degree* 1 *on* $\mathsf{A}(\tilde{\mathsf{L}}\mathbf{Z})$ *are cohomology operators for all* $t \in \mathbb{R}$, *i.e.* $(\partial^t_{\mathcal{J}})^2 = 0$.

Proof. The implication "if" is essentially the graded Jacobi identity applied to the bracketing with canonical structures. The other follows from the fact that the corresponding Schouten and Schouten–Jacobi brackets have no central elements among 3-tensors. □

The cohomology associated to ∂_Λ we will call the *aff-Poisson cohomology*. The cohomology associated to $\partial^0_{\mathcal{J}}$ (resp. $\partial^1_{\mathcal{J}}$) we will call *aff-Jacobi cohomology* (resp., *aff-Lichnerowicz-Jacobi cohomology*).

For any Lie algebroid structure on a vector bundle E denote by $£_Y$ the corresponding Lie differential $£_Y = i_Y \circ \mathrm{d}_E - (-1)^{|Y|}\mathrm{d}_E \circ i_Y$ with respect to the multisection $Y \in \mathsf{Sec}(\wedge^{|Y|} E)$. Here and further $|Y|$ denotes the degree of the tensor Y.

Theorem 27.

(a) $\Lambda \in \mathsf{Sec}(\wedge^2 \tilde{\mathsf{T}}\mathbf{Z})$ *represents an aff-Poisson structure on* $\mathbf{Z}$ *if and only if the Lie differential*

$$£_\Lambda = i_\Lambda \circ \mathrm{d}_{\tilde{\mathsf{T}}\mathbf{Z}} - \mathrm{d}_{\tilde{\mathsf{T}}\mathbf{Z}} \circ i_\Lambda,$$

which is a graded operator of degree -1 *on* $\mathsf{A}(\tilde{\mathsf{T}}^*\mathbf{Z})$, *is a homology operator, i.e.* $(£_\Lambda)^2 = 0$.

(b) $\mathcal{J} \in \mathsf{Sec}(\wedge^2 \tilde{\mathsf{L}}\mathbf{Z})$ *represents an aff-Jacobi structure on* $\mathbf{Z}$ *if and only if the Jacobi-Lie differential*

$$£^{\phi^0,t}_{\mathcal{J}}(\omega) = £_{\mathcal{J}}(\omega) + (|\omega| + t) i_{i_{\phi^0}\mathcal{J}}(\omega) + \phi^0 \wedge i_{\mathcal{J}}(\omega),$$

which is a graded operator of degree -1 *on* $\mathsf{A}(\tilde{\mathsf{L}}^*\mathbf{Z})$, *is a homology operator for each* $t \in \mathbb{R}$, *i.e.* $(£^{\phi^0,t}_{\mathcal{J}})^2 = 0$.

Proof. The part "if" follows from the identities (see [5])

$$2(£_\Lambda)^2 = -£_{[\![\Lambda,\Lambda]\!]_{\tilde{\mathsf{T}}\mathbf{Z}}}$$

and

$$2(£^{\phi^0,t}_{\mathcal{J}})^2 = -£^{\phi^0,t}_{[\![\mathcal{J},\mathcal{J}]\!]^{\phi^0}_{\tilde{\mathsf{L}}\mathbf{Z}}}.$$

The other part follows from the fact that passing to the Lie differentials in the algebroids in question is injective for 3-tensors. □

The homology associated with $£_\Lambda$ we will call *aff-Poisson homology*. The homology associated with $£_{\mathcal{J}}^{\phi^0,0}$ we will call *aff-Jacobi homology*.

Aff-Poisson and aff-Jacobi structures give also rise to the corresponding triangular Lie bialgebroids and Jacobi bialgebroids (cf. [4,10,18]). We will not go into the details here.

14. Applications

Example 9 (Tulczyjew and Urbański [22,25]). In gauge theories potentials are interpreted as connections on principal bundles. In the electrodynamics the gauge group is $(\mathbb{R}, +)$ and the potential is a connection on a principal bundle $\zeta : \mathbf{Z} \to M$ over the space–time M, i.e. on an AV-bundle $\mathbf{Z} = (Z, 1_M)$ over M. An electromagnetic potential is a section $\alpha : M \to \mathsf{P}\mathbf{Z}$.

According to [27], the phase manifold for a particle with the charge $e \in \mathbb{R}$ is obtained by the symplectic reduction of $\mathsf{T}^*\mathbf{Z}$ with respect to the coisotropic submanifold

$$K_e = \{p \in \mathsf{T}^*\mathbf{Z} : \langle p, X_{\mathbf{Z}}\rangle = -e\}.$$

Let us denote by $\mathsf{P}_e\mathbf{Z}$ the reduced phase space. It is easy to see that it is an affine bundle modeled on T^*M. We show that $\mathsf{P}_e\mathbf{Z}$ is the phase bundle for certain special affine bundle $\mathbf{Z}_e$.

First, let $\mathbf{Y} = Z\times^a\mathbf{I}$ be the trivial AV-bundle over Z. We define an $\mathbb{R}$-action on $\mathbf{Y}$ by the formula

$$(Z \times \mathbb{R}) \times \mathbb{R} \ni ((z, r), t) \mapsto (z + t, r + te) \in Z \times \mathbb{R} = Y.$$

The space of orbits is an affine bundle modeled on $M \times \mathbb{R}$ and denoted by Z_e. We denote by ζ_e the canonical projection $Z_e \to M$. The distinguished section of $\mathsf{V}(\mathbf{Y})$ (the function 1_Z) projects to the constant function 1_M and the canonical projection $\lambda_e : Y \to Z_e$ is a morphism of special affine bundles $\mathbf{Y} \to \mathbf{Z}_e = (Z_e, 1_M)$. The induced $\mathbb{R}$-action on $\mathbf{Z}_e$ has the form

$$\lambda_e(z, r) + s = \lambda_e(z, r + s) = \lambda_e(z + t, r + s + te).$$

For $e = 0$ the bundle $\mathbf{Z}_e$ is trivial: $\mathbf{Z}_0 = M \times \mathbb{R}$ and for $e \neq 0$ we have a diffeomorphism

$$\Phi_e : Z \to Z_e, \qquad : z \mapsto \lambda_e(z, 0).$$

The diffeomorphism Φ_e is an isomorphism of the special affine bundle $(Z, -(1/e)1_M)$ onto $\mathbf{Z}_e$:

$$\Phi_e\left(z - \frac{1}{e}r\right) = \lambda_e\left(z - \frac{1}{e}r, 0\right) = \lambda_e(z, r) = \lambda_e(z, 0) + r = \Phi_e(z) + r.$$

In particular, $\mathbf{Z}_{-1} = \mathbf{Z}$ and $\mathbf{Z}_1 = \bar{\mathbf{Z}}$. To put it simpler, let us observe that, according to [2, Example 3], $\mathbf{Z}_e$ is just the level-set of $1_{\mathbf{Z}}$ in $\hat{\mathbf{Z}}$ associated with value $-e$. The diffeomorphism Φ_e comes just from the homotety by $-e$ in $\hat{\mathbf{Z}}$.

Let σ be a section of ζ_e. The function $\lambda_e^*\sigma$ on $\mathbf{Z}$ has the property

$$X_{\mathbf{Z}}(\lambda_e^*\sigma) = -e.$$

We conclude that the induced by λ_e relation $\mathsf{P}Y \to \mathsf{P}Z_e$ is the symplectic reduction with respect to a coisotropic submanifold

$$K_e = \{p \in \mathsf{T}^*\mathbf{Z} : \langle p, X_{\mathbf{Z}}\rangle = -e\}.$$

Thus the phase manifold $\mathsf{P}_e\mathbf{Z}$ for a particle with the charge e is the phase bundle for the special affine bundle $\mathbf{Z}_e$. Another way to see this is to use the decomposition $\mathsf{T}^*\mathbf{Z} = \tilde{\mathsf{T}}^*\mathbf{Z} \times_M \mathbf{Z}$. The symplectic reduction in question is the reduction with respect to the moment map for the phase lift of the canonical $\mathbb{R}$-action on $\mathbf{Z}$, i.e.

$$\mathsf{P}_e\mathbf{Z} = \{[\alpha_{z_m}] \in \tilde{\mathsf{T}}^*\mathbf{Z} : \langle \alpha_{z_m}, X_{\mathbf{Z}}(z_m)\rangle_{z_m} = -e\}.$$

But $\langle \alpha_{z_m}, X_{\mathbf{Z}}(z_m)\rangle_{z_m} = -e$ is equivalent to $\langle \alpha_{z_m}, -(1/e)X_{\mathbf{Z}}(z_m)\rangle_{z_m} = 1$ that is a form of a definition of $\mathsf{P}\mathbf{Z}_e$. That the symplectic structure on $\mathsf{P}\mathbf{Z}$, defined originally as the pull-back from T^*M when a section of $\mathbf{Z}$ is chosen, coincides with the one reduced from $\mathsf{T}^*\mathbf{Z}$ can be easily checked in the given trivialization.

The isomorphism Φ_e gives a one-to-one correspondence between sections of ζ and sections of ζ_e, for $e \neq 0$. It follows that a chosen section of ζ provides a trivialization of $\mathbf{Z}$ and also of $\mathbf{Z}_e$. In such trivializations, a section σ of ζ and the corresponding section $\Phi_e \circ \sigma$ of ζ_e are functions on M related by the formula

$$\Phi_e \circ \sigma(m) = -e\sigma(m).$$

The correspondence $\sigma \to \Phi_e \circ \sigma$ of sections projects to a correspondence of affine covectors and consequently gives a correspondence of affine one-forms. Let α be a section of $\mathsf{P}\zeta : \mathsf{P}\mathbf{Z} \to M$ and α_e be the corresponding section of $\mathsf{P}\zeta_e$. In a given trivialization, the sections α and α_e are one-forms related by the formula $\alpha_e = -e\alpha$.

The Lagrangian of a relativistic charged particle is a section L_e of the bundle $\tilde{\mathsf{T}}\zeta_a : \tilde{\mathsf{T}}\mathbf{Z}_e \to \mathsf{T}M$ over the open set $C = \{v \in \mathsf{T}M : g(v, v) > 0\}$ given by the formula

$$L_e(v) = \langle \alpha_e, v\rangle + m\sqrt{g(v, v)},$$

where g is the metric tensor on the space–time M, m is the mass of the particle, and $\langle \alpha_e, v\rangle = \alpha_e(v)$, where an element of $\mathsf{P}\mathbf{Z}_e$ is interpreted as a linear section of $\tilde{\mathsf{T}}\xi_e : \tilde{\mathsf{T}}\mathbf{Z}_e \to \mathsf{T}M$, i.e. as an element of $\mathsf{LS}(\tilde{\mathsf{T}}\mathbf{Z}_e)$. In this example Lagrangians are sections of an AV-bundle. Hamiltonians are ordinary functions but not on a cotangent bundle but on the affine phase bundle $\mathsf{P}\mathbf{Z}_e$.

Example 10 (Urbański, cf. [24]). The space of events for the inhomogenous formulation of time-dependent mechanics is the space–time M fibrated over the time $\mathbb{R}$. First-jets of this fibration form the infinitesimal (dynamical) configuration space. Since there is the distinguished vector field ∂_t on $\mathbb{R}$, the first-jets of the fibration over time can be identified with those vectors tangent to M which project on ∂_t. Such vectors form an affine subbundle A of the tangent bundle $\mathsf{T}M$ modeled on the bundle $\mathsf{V}M$ of vertical vectors. The Lagrange formalism in the affine formulation originates on the AV-bundle $A\times^a\mathbf{I} \rightarrow A$ and the Lagrangians are ordinary functions on A. The Hamilton formalism now takes place not on the dual vector bundle V^*M of $\mathsf{V}M$, as in the classical approach, but on the dual AV-bundle $\zeta : (A\times^a\mathbf{I})^\# = \mathbf{A}^\dagger \rightarrow \mathsf{V}^*M$ which can be recognized as $\zeta : \mathsf{T}^*M \rightarrow \mathsf{T}^*M/\langle \mathrm{d}t\rangle$ and which carries a canonical aff-Poisson structure induced from the canonical symplectic Poisson bracket on T^*M (cf. Example 5). The Hamiltonians are sections of this bundle. To compare with the standard approach, let us assume that we have a decomposition $M = Q \times \mathbb{R}$ of the space–time into a product of space and time. This induces the decomposition $\mathsf{T}^*M = \mathsf{T}^*Q\times\mathsf{T}^*\mathbb{R}$. Sections σ of ζ can be identified with functions (time-dependent Hamiltonians) $H = H(\alpha, t)$ on $\mathsf{V}^*M = \mathsf{T}^*Q\times\mathbb{R}$ by $\sigma_H(\alpha, t) = (\alpha, t, -H(\alpha, t)\,\mathrm{d}t)$. The dynamics induced by the section σ_H is, as in Example 5, the projection of the dynamics on T^*M induced by $\mathbf{F}_{\sigma_H}$. The distinguished section of T^*M is $\mathrm{d}t$, so that the distinguished section in the AV-bundle ζ is represented by $-\mathrm{d}t$. Thus,

$$\mathbf{F}_{\sigma_H}(\alpha, t, p) = H(\alpha, t) + p,$$

where (t, p) are the standard Darboux coordinates in $\mathsf{T}^*\mathbb{R}$. The Hamiltonian vector field of $\mathbf{F}_{\sigma_H}$ on T^*M is therefore $X_{H_t} + \partial_t$, where X_{H_t} is the Hamiltonian vector field of $H_t(x) = H(x, t)$ on T^*Q, so we have recovered the correct dynamics. However, in our picture, the term ∂_t is not added 'by hand' but it is generated from σ_H by means of the aff-Poisson structure. Of course, if we have no decomposition into space and time, there is no canonical ∂_t on M and nothing canonical can be added by hand in the standard approach. This problem disappears in the aff-Poisson formulation. In this example, Hamiltonians are sections of an AV-bundle and Lagrangians are ordinary functions however not on a vector but on an affine bundle.

Example 11. The last example is devoted to a Hamiltonian formulation of dynamics of one massive particle in the Newtonian space–time (cf. [6,11]). Even in a fixed inertial frame, up to now, there was no satisfactory description of the dynamics in the Hamiltonian formulation. First, we would like to present difficulties that appear while constructing the description for the dynamics in an inertial frame and then we will show the solution in the language of AV-geometry.

Let N be the Newtonian space–time i.e. a four-dimensional affine space equipped with a covector τ being an element of the dual of the model vector space $\mathsf{V}(N)$ and an euclidean metrics g on the kernel of τ. The covector τ is used for measuring time intervals between events and the metrics measures spatial distance between simultaneous events. We will denote the kernel of τ by E_0 and the level-1 set of τ by E_1. The vector space E_0 is of course a vector subspace of $\mathsf{V}(N)$ and E_1 is an affine subspace of $\mathsf{V}(N)$ modeled on E_0. The elements of E_1 are physical velocities of particles. On the other hand, every element of E_1 represents a class of inertial observers moving in the space–time with the same constant velocity. Such class of observers will be called an inertial frame. The configuration space for one massive particle is $N \times E_1$. Having an inertial frame u, we can identify the affine subspace E_1 with its model vector space, the phase space is generally accepted to be $N\times E_0^*$. The correct phase equations for the potential φ are:

$$\dot{x} = g^{-1}\left(\frac{p}{m}\right) + u, \tag{41}$$

$$\dot{p} = -\mathrm{d}_s\varphi(x), \tag{42}$$

where $(x, p, \dot{x}, \dot{p})$ is an element of $\mathsf{T}(N\times E_0^*)$ that can be identified with $N\times E_0^*\times\mathsf{V}(N)\times E_0^*$. The subscript in d_s means that we differentiate only in the spatial directions, i.e. vertical with respect to the projection on time.

The standard Hamiltonian description is based on the fact that $N \times E_0^*$ is a Poisson manifold with the Poisson structure being reduced from the canonical Poisson structure of $\mathsf{T}^*N \simeq N \times \mathsf{V}(N)^*$. The problem is that from the Hamiltonian

$$h_u(x, p) = \frac{1}{2m}\langle p, g^{-1}(p)\rangle + \varphi(x)$$

we obtain the vector field which is vertical with respect to the projection on time:

$$\dot{x} = g^{-1}\left(\frac{p}{m}\right), \qquad \dot{p} = -\mathrm{d}_s\varphi(x).$$

Any vertical vector field cannot be a physical motion, so we have to add 'by hand' the constant vector field u. As in the previous example, this problem can be solved by replacing the Poisson structure on the phase manifold by an affine Poisson structure. However, the equations of motion as well as the affine Poisson structure depend on the choice of the reference frame.

To get frame-independent formulation for the dynamics, let us consider first frame-dependent Lagrangian ℓ_u. It is a function defined on $N \times E_1$

$$\ell_u(x, v) = \tfrac{1}{2}m\langle g(v-u), v-u\rangle - \varphi(x).$$

Let us look at the solution of this problem. If u and u' are two inertial frames then the difference

$$f_{u,u'}(v) = \ell_u(x, v) - \ell_{u'}(x, v) = m\left\langle g(u'-u), v - \tfrac{1}{2}(u+u')\right\rangle$$

is an affine function on E_1. Now we define an equivalence relation $\sim_\ell$ in the set $E_1\times E_1\times\mathbb{R}$ by

$$(u, v, r) \sim_\ell (u', v', r') \leftrightarrow v = v', \quad r = r' + f_{u,u'}(v).$$

The set of equivalence classes for $\sim_\ell$ will be denoted by A_0. We observe that since $f_{u,u'}$ is an affine function, A_0 is an affine space of dimension 4. There is a projection from A_0 to E_1. The model vector space for A_0 is $(E_1 \times E_0 \times \mathbb{R})\backslash \sim_{v\ell}$, where the equivalence relation $\sim_{v\ell}$ is in a sense the linear part of $\sim_\ell$: we say that two elements (u, w, r) and (u', w', r') of $E_1 \times E_0 \times \mathbb{R}$ are equivalent if

$$w = w', \qquad r = r' + m\langle g(u-u'), w\rangle.$$

In $\mathsf{V}(A_0)$ we distinguish an element $w_0 = [u, 0, 1]$ so now $\mathbf{A}_0 = (A_0, w_0)$ is a special affine space and $\mathbf{A} = N \times \mathbf{A}_0$ is a special affine bundle over N. The mapping

$$(x, v) \mapsto (x, [u, v, \ell_u(x, v)])$$

is a section of $\mathsf{AV}(\mathbf{A})$ and can be understood as frame-independent Lagrangian. Note that it is no longer a function but a section of an AV-bundle. Any section of $\mathbf{A}$ can be represented in the form

$$x \mapsto \tilde{X}(x) = [u, X(x), r(x)],$$

where X is a vector field on N with values in E_1 and r is a function on N. We define a bracket on sections of $\mathbf{A}$ by the following formula

$$[\tilde{X}, \tilde{Y}] = [u, [X, Y], Xs - Yr],$$

where $\tilde{X}(x) = [u, X(x), r(x)]$ and $\tilde{Y}(x) = [u, Y(x), s(x)]$. The definition is correct. Indeed, if we have other representatives $X(x) = [u', X(x), r(x) - f_{u,u'}(X(x))]$ and $Y(x) = [u', Y(x), s(x) - f_{u,u'}(Y(x))]$, then, since $f_{u,u'}$ is affine, $(\pounds_{X(x)}(f_{u,u'} \circ Y))(x) = (f_{u,u})_{\mathsf{V}} \circ (\pounds_{X(x)}Y)(x)$, where $\pounds_{X(x)}$ is the directional derivative in the direction $X(x)$ and $(f_{u,u'})_{\mathsf{V}}$ is the vector part of $f_{u,u'}$. Moreover, $(\pounds_{X(x)}Y - \pounds_{Y(x)}X)(x) = [X, Y](x)$, so that we get

$$X(r - f_{u,u'} \circ Y) - Y(s - f_{u,u'} \circ X) = X(r) - Y(s) - (f_{u,u'})_{\mathsf{V}} \circ [X, Y],$$

that proves the correctness of the definition.

Having two vector fields X, Y with values in E_1 we have

$$0 = (\mathrm{d}\tau)(X, Y) = X\langle \tau, Y\rangle - Y\langle \tau, X\rangle + \langle \tau, [X, Y]\rangle.$$

Since $X\langle \tau, Y\rangle = Y\langle \tau, X\rangle = 0$ we obtain that $\langle \tau, [X, Y]\rangle = 0$, i.e. $[X, Y] \in E_0$. The bracket of two sections of $\mathbf{A}$ is therefore a section of $\mathsf{V}(\mathbf{A})$ and it is easy to see that it is a Lie affgebroid bracket with the anchor morphism $\gamma : \mathcal{A} \to \mathsf{T}N$ defined as $\gamma([u, X, r]) = X$. Moreover, the section w_0 is central for the bracket, i.e. $[\tilde{X}, w_0]^2_{\mathsf{V}} = 0$ for all X. Therefore, according to the Theorem 23, we have that the corresponding aff-Jacobi bracket on the AV-bundle $\mathsf{AV}(\mathbf{A}^{\#})$ is aff-Poisson. We claim that this structure is the correct structure for generating the equations of motion for the Hamiltonian formulation of the dynamics in question.

Indeed, $\mathbf{A}^{\#}$ is by definition $\mathbf{Aff}(\mathbf{A}, \mathbf{I})$. Like in the Lagrangian case, we will represent it as the set of cosets of an appropriate equivalence relation. In the space $E_1 \times E_0^* \times \mathbb{R}$ we define an equivalence relation $\sim_h$ by

$$(u, p, s) \sim_h (u', p', s') \leftrightarrow p = p' + mg(u - u'),$$
$$s = s' + \langle p, u - u'\rangle + \tfrac{1}{2}m\langle g(u - u'), u - u'\rangle.$$

An equivalence class $[u, p, s]$ represents an affine function $\xi_{[u,p,s]}$ on A_0 given by

$$\xi_{[u,p,s]}([u, v, r]) = \langle p, v - u\rangle + s - r.$$

Its linear part $(\xi_{[u,p,s]})_{\mathsf{V}}([u, w, r]) = \langle p, w\rangle - r$ gives -1 while evaluated on w_0. The model vector space for $\mathbf{A}_0^{\#}$ is $\mathsf{V}(N)^*$ with distinguished element τ. We have:

$$[u, p, s] + \pi = [u, p + \iota(\pi), s + \langle \pi, u\rangle],$$

where ι is the canonical projection from $\mathsf{V}(N)^*$ on E_0^*. Let us denote by P the space of affine momenta, i.e. the space $E_0 \times E_0^*/\sim_P$ for the relation

$$(u, p) \sim_P (u', p') \leftrightarrow p = p' + mg(u - u').$$

We observe that $\mathsf{AV}(\mathbf{A}^\#)$ is an AV-bundle over $N \times P$. The frame-independent Hamiltonian is a section of $\mathsf{AV}(\mathbf{A}^\#)$:

$$(x, [u, p]) \mapsto h((x, [u, p]) = (x, [u, p, h_u(x, p)]).$$

Using the canonical aff-Poisson structure on $\mathsf{AV}(\mathbf{A}^\#)$ we can generate out of h an affine derivation of $\mathsf{AV}(\mathbf{A}^\#)$, i.e. the section of $\tilde{\mathsf{T}}(\mathsf{AV}(\mathbf{A}^\#))$. This section projects to a vector field on $N \times P$ that is understood as the equation of motion.

Now, let us calculate the equations of motion in coordinates. For, we choose an inertial frame u and the coordinates (x^0, x^i), $i = 1, 2, 3$ such that $\partial_0 = u$. By (p_i) we denote the adapted coordinates on E_0^*. Using the inertial frame we have the following identifications:

$$\mathbf{A}_0 \simeq E_1 \times \mathbf{I}, \qquad \mathbf{A} \simeq N \times E_1 \times \mathbf{I}, \qquad \mathbf{A}^\# \simeq N \times E_0^* \times \mathbf{I},$$
$$\mathsf{Sec}(\mathsf{AV}(\mathbf{A}^\#)) \simeq \mathcal{C}^\infty(N \times E_0^*).$$

The bracket of sections of $N \times E_1 \times \mathbb{R}$, has the obvious form $[(X, \xi), (Y, \vartheta)] = ([X, Y], X\vartheta - Y\xi)$. If the sections take values in E_1 then the bracket takes values in E_0. The affine function on $N \times E_0^*$ that corresponds to the section (X, ξ) is

$$\iota_{(X,\xi)}(x, p) = \langle p, X\rangle - \xi.$$

The Poisson bracket for functions corresponding to sections (X, ξ), (Y, ϑ) is given by the formula

$$\{\iota_{(X,\xi)}, \iota_{(Y,\vartheta)}\} = \iota_{[(X,\xi),(Y,\vartheta)]} = \langle p, [X, Y]\rangle - X\vartheta + Y\xi,$$

which in coordinates reads

$$\begin{aligned}\{\iota_{(X,\xi)}, \iota_{(Y,\vartheta)}\} &= p_i X^j \partial_j Y^i - p_i Y^j \partial_j X^i + p_i \partial_0 Y^i - p_i \partial_0 X^i - X^i \partial_i \vartheta \\ &\quad - \partial_0 \vartheta + Y^i \partial_i \xi + \partial_0 \xi.\end{aligned}$$

From the above formula we obtain that

$$\{h, \cdot\} = (\partial^j h)\partial_j + \partial_0 - (\partial_j h)\partial^j - \partial_0 h.$$

The vector part of the above operator is exactly what we had in (41) and (42). In this example both, Hamiltonians and Lagrangians are sections of AV-bundles and not ordinary functions.

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